Lecture 27

0. Recall - from last lecture - to a Cartan matrix $A = (a_{ij})_{i,j\in I}$ and symmetrizing integers $(d_i)_{i\in I}$ - (so DA is symmetric) we associated a Hopf algebra $\widetilde{\mathcal{U}}_\hbar$. We introduced Hopf subalgebras $\widetilde{\mathcal{U}}^{\geq 0}, \widetilde{\mathcal{U}}^{\leq 0}$; a bilinear form $\widetilde{\mathcal{U}}^{\leq 0} \times \widetilde{\mathcal{U}}^{\geq 0} \to \mathbb{C}((\hbar))$

$\to$ Hopf ideals $\mathrm{rad}^{\geq 0} \subset \widetilde{\mathcal{U}}^{\geq 0}$ defined by $\{x \in \widetilde{\mathcal{U}}^{\geq 0} \mid (y,x) = 0\ \forall y \in \widetilde{\mathcal{U}}^{\leq 0}\}$.

$\mathrm{rad}^{\leq 0} \subset \widetilde{\mathcal{U}}^{\leq 0}$

We showed that $\underbrace{(\mathrm{rad}^{\leq 0} \cdot \widetilde{\mathcal{U}}^{\geq 0} + \widetilde{\mathcal{U}}^{\leq 0} \cdot \mathrm{rad}^{\geq 0})}_{\mathrm{rad}} \subset \widetilde{\mathcal{U}}$ is a 2-sided ideal

$$\mathcal{U}_\hbar(\mathfrak{g}) := \widetilde{\mathcal{U}}_\hbar / \mathrm{rad}$$

- quantum group assoc. to A (& so also $\mathfrak{g}$ = simple Lie alg assoc. to A.)

$\boxed{\text{Serre Rel}^{\text{n}}\text{s}}$

and $\forall\, i \neq j \in I$; $\theta^+_{ij} = \sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} E_i^{m-s} E_j E_i^s \in \mathrm{rad}^{\geq 0}$ (– for F's $\in \mathrm{rad}^{\leq 0}$)

(here: $m = 1 - a_{ij}$; $[\ell]_{q_i} = \dfrac{q_i^\ell - q_i^{-\ell}}{q_i - q_i^{-1}}$; $\begin{bmatrix} m \\ \ell \end{bmatrix} = \dfrac{[m][m-1]\ldots[m-\ell+1]}{[\ell]!}$ for $m \in \mathbb{Z}$, $\ell \in \mathbb{Z}_{\geq 0}$)

1. Remark on Serre relations.

Let V be a f.d. vector space and $\rho: \widetilde{\mathcal{U}}_\hbar \to \mathrm{End}(V)[[\hbar]]$ an algebra hom. (over $\mathbb{C}[[\hbar]]$). Then $\rho(\theta^\pm_{ij}) = 0$ $\forall\, i \neq j$.

Proof. - Consider $\mathcal{U}_{dih}(sl_2)$ action on $\mathrm{End}(V)[\![\hbar]\!]$.

(For - case)

$x \in \mathcal{U}_{dih}(sl_2) \rightsquigarrow \varphi(x)\cdot A : \begin{cases} \varphi(E_i)\cdot A = \rho(K_i)^{-1}[A, \rho(E_i)] \\ \varphi(F_i)\cdot A = A\rho(F_i) - \\ \qquad \rho(F_i)\rho(K_i)A\rho(K_i)^{-1}. \end{cases}$

$A \in \mathrm{End}(V)[\![\hbar]\!]$

$$\varphi(K_i)\cdot A = \rho(K_i)A\rho(K_i)^{-1}$$

(Ex: Check that it is an action).

Take $A = \rho(F_j) \in \mathrm{End}(V)[\![\hbar]\!]$. $(j \neq i)$:

$$\varphi(H_i)\cdot A = -a_{ij}A \quad ; \quad \varphi(E_i)\cdot A = 0$$

$\Rightarrow \varphi(F_i)^{1-a_{ij}}\cdot A = 0$. The assertion follows from $l = 1-a_{ij}$ case of the following claim:

(Finite-dim'l repn theory)

Claim: $\varphi(F_i)^{l}\cdot A = \rho\left(\sum_{k=0}^{l}(-1)^{l-k} q_i^{(l-k)(1-a_{ij}-l)} \begin{bmatrix} l \\ k \end{bmatrix}_{q_i} F_i^{l-k}F_jF_i^{k}\right)$

(I am dropping ρ now.)

$$l=1: \varphi(F_i)\cdot F_j = F_jF_i - F_iK_iF_jK_i^{-1}$$
$$= -q_i^{-a_{ij}}F_iF_j + F_jF_i \quad \checkmark$$

$l+1$ - induction step - $F_i\cdot(F_i^{l}\cdot A)$

$$= F_i\cdot\left(\sum_{k=0}^{l}(-1)^{l-k} q_i^{(l-k)(1-a_{ij}-l)} \begin{bmatrix} l \\ k \end{bmatrix}_{q_i} F_i^{l-k}F_jF_i^{k}\right)$$

$$= \sum_{k=0}^{\ell} (-1)^{\ell-k} q_i^{(\ell-k)(1-a_{ij}-\ell)} \begin{bmatrix} \ell \\ k \end{bmatrix}_{q_i} \Big(\big(F_i^{\ell-k} F_j F_i^{k}\big) F_i - F_i \, q_i^{-2\ell-a_{ij}} \big(F_i^{\ell-k} F_j F_i^{k}\big) \Big)$$

Coeff of $F_i^{\ell+1-t} F_j F_i^{t}$

$t=0$: $\quad -(-1)^{\ell} q_i^{\ell(1-a_{ij}-\ell)-2\ell-a_{ij}}$

$$= (-1)^{\ell+1} q_i^{-\ell-(\ell+1)a_{ij}-\ell^2}$$

$$= (-1)^{\ell+1} q_i^{(\ell+1)(-\ell-a_{ij})} \quad \checkmark$$

(Similarly $t=\ell+1$ case)

<u>$1 \leq t \leq \ell$</u>:

$$-(-1)^{\ell-t} q_i^{(\ell-t)(1-a_{ij}-\ell)-2\ell-a_{ij}} \begin{bmatrix} \ell \\ t \end{bmatrix}_{q_i}$$

$$+ (-1)^{\ell-t+1} q_i^{(\ell-t+1)(1-a_{ij}-\ell)} \begin{bmatrix} \ell \\ t-1 \end{bmatrix}_{q_i}$$

$$= (-1)^{\ell+1-t} q_i^{(\ell+1-t)(\cancel{1}-a_{ij}-\ell)} \Bigg[\underbrace{q_i^{-t} \begin{bmatrix} \ell \\ t \end{bmatrix}_{q_i} + q_i^{\ell+t+1} \begin{bmatrix} \ell \\ t-1 \end{bmatrix}_{q_i}}_{\substack{\text{Pascal's id: (q-version)} \\ \begin{bmatrix} \ell+1 \\ t \end{bmatrix}_{q_i}}} \Bigg]$$

$\square$

2. $U_\hbar(\mathfrak{g})$ infact has the expected presentation -

Generators : $\quad H \in \mathfrak{h}$; $\{E_i, F_i\}_{i \in I}$

Relations : $\quad [H, H'] = 0 \quad \forall\, H, H' \in \mathfrak{h}$;

- $[H, E_i] = \alpha_i(H) E_i$; $[H, F_i] = -\alpha_i(H) F_i$ $\forall H \in \mathfrak{h}$; $i \in I$.

- $$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \qquad \forall i, j \in I$$

- $$\forall i \neq j \in I : \quad \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} E_i^{1-a_{ij}-s} E_j E_i^s = 0 = \left(\text{same w/ } F\text{'s}\right).$$

This result is proved using an analogue of our old Weyl-group action.

3. Quantum Weyl group action - sl_2 case. $\left(U_h(sl_2) = \langle E, F, H \rangle / \ldots \text{rel}^{\text{n}}\text{s}\right)$.

Definition. $$\mathcal{S} = \exp_{q^{-1}}(q^{-1} E K^{-1}) \exp_{q^{-1}}(-F) \exp_{q^{-1}}(q E K) \cdot q^{\frac{H(H+1)}{2}}$$

— recall $$\exp_q(x) = \sum_{n \geq 0} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]!}.$$

This element defines an invertible operator on representations of $U_h(sl_2)$ - where E & F act locally nilpotently (& H-diagonally)

Recall: L_λ ($\lambda \in \mathbb{Z}_{\geq 0}$) is $(\lambda+1)$-dim'l irred. repn. - basis $\{v_0, \ldots, v_\lambda\}$

$$H . v_r = (\lambda - 2r) v_r$$

$$E v_r = [\lambda - r + 1] v_{r-1} \quad ; \quad F v_r = [r+1] v_{r+1}.$$

Lemma. $\mathcal{S}\, v_r = (-1)^{\lambda-r} q^{(\lambda-r)(r+1)}\, v_{\lambda-r}$.

Proof: (Highly computational - uses the following identities of q-binomials:

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{[m]\,[m-1]\cdots[m-n+1]}{[n]!} \qquad \forall\ m \in \mathbb{Z}\ ;\ n \in \mathbb{Z}_{\geq 0}.$$

(1) $$\begin{bmatrix} m \\ n \end{bmatrix} = (-1)^n \begin{bmatrix} n-1-m \\ n \end{bmatrix}$$

(2) $$q^{-l} \begin{bmatrix} m \\ l \end{bmatrix} + q^{m-l+1} \begin{bmatrix} m \\ l-1 \end{bmatrix} = \begin{bmatrix} m+1 \\ l \end{bmatrix}$$

(use (2) to prove): (3) $$\sum_{i=0}^{n} q^{ai-b(n-i)} \begin{bmatrix} a \\ n-i \end{bmatrix} \begin{bmatrix} b \\ i \end{bmatrix} = \begin{bmatrix} a+b \\ n \end{bmatrix}.\ )$$

(relatively) Easy calculation shows: coeff of $v_{\lambda-t}$ in $\mathcal{S}\, v_r$ is given by

$$q^{\frac{(\lambda-2r)(\lambda-2r+1)}{2}} \cdot \sum_{\substack{a,b,c \in \mathbb{Z}_{\geq 0} \\ b-a-c = \lambda-(r+t)}} (-1)^b\, q^{\frac{a(a+1)}{2} - \frac{b(b-1)}{2} + \frac{c(c+1)}{2} + c(\lambda-2r) + a(\lambda-2t)} \begin{bmatrix} t \\ a \end{bmatrix} \begin{bmatrix} r-c+b \\ b \end{bmatrix} \begin{bmatrix} \lambda-r+c \\ c \end{bmatrix}.$$

$\boxed{\text{get rid of } b}$

$$= \sum_a (-1)^{\lambda-r+a-t} \begin{bmatrix} t \\ a \end{bmatrix} q^{a(r-t+1)+\lambda(t+1)-rt+\frac{r(r-1)}{2}-\frac{t(t+1)}{2}} \cdot \sum_c (-1)^c\, q^{(r-c)(r-\lambda-1)-c(\lambda-t+a)} \begin{bmatrix} \lambda-r+c \\ c \end{bmatrix} \overbrace{\begin{bmatrix} \lambda-t+a \\ r-c \end{bmatrix}}^{\begin{bmatrix} r-c+b \\ b \end{bmatrix}}$$

$$\text{Last sum} = \sum_c (-1)^c q^{(r-c)(r-\lambda-1)-c(\lambda-t+a)} \begin{bmatrix}\lambda-r+c\\ c\end{bmatrix}\begin{bmatrix}\lambda-t+a\\ r-c\end{bmatrix}$$

$$= \sum_c q^{(r-c)(r-\lambda-1)-c(\lambda-t+a)} \begin{bmatrix}r-\lambda-1\\ c\end{bmatrix}\begin{bmatrix}\lambda-t+a\\ r-c\end{bmatrix}$$ (using eq^n (1) from the last page)

$$\underset{\substack{\uparrow\\ \text{using (3)}}}{=} \begin{bmatrix}r-t+a-1\\ r\end{bmatrix} = (-1)^r \begin{bmatrix}t-a\\ r\end{bmatrix} = (-1)^{t-a}\begin{bmatrix}-r-1\\ t-a-r\end{bmatrix}$$

$$\Rightarrow \langle v_{\lambda-t} | \mathcal{S} | v_r\rangle = \sum_a (-1)^{\lambda-r} q^{\lambda(t+1)-rt+\frac{r(r-1)}{2}-\frac{t(t+1)}{2}-t(t-r)} .$$

(coeff of $v_{\lambda-t}$ in $\mathcal{S}.v_r$)

$$\underbrace{\cdot\, q^{t(t-r-a)+a(r+1)} \begin{bmatrix}t\\ a\end{bmatrix}\begin{bmatrix}-r-1\\ t-a-r\end{bmatrix}}_{\xrightarrow{(3)\text{ again}} \begin{bmatrix}t-r-1\\ t-r\end{bmatrix} = \delta_{t,r}}$$

$$= \delta_{t,r} (-1)^{\lambda-r} q^{\lambda(r+1)-r^2-r} = \delta_{t,r} (-1)^{\lambda-r} q^{(\lambda-r)(r+1)} . \qquad \square$$

Remark. - The proof of this lemma shows that

$$\mathcal{S}.v_r = \sum_{\substack{a,b,c\geq 0\\ b-a-c=\lambda-2r}} (-1)^b q^{b-ac} \frac{E^a}{[a]!}\frac{F^b}{[b]!}\frac{E^c}{[c]!}.v_r$$

4. Automorphism of $\mathcal{U}_\hbar(sl_2)$. - Let $T: \mathcal{U}_\hbar(sl_2) \to \mathcal{U}_\hbar(sl_2)$ be given by

$$T(H) = -H \;;\; T(E) = -FK \;;\; T(F) = -K^{-1}E.$$

Then, we have the following ~~lemma~~ corollary of Lemma 3 above:

Cor. For every $u \in \mathcal{U}_\hbar(sl_2)$ and $v \in L_\lambda$, we have

$$\mathcal{S}\cdot(u\cdot v) = T(u)\cdot(\mathcal{S}\, v).$$

Remark - We view this corollary as saying: $\boxed{\mathcal{S}\, u\, \mathcal{S}^{-1} = T(u)}$.

5. Arbitrary $\mathfrak{g}$ - For each $i \in I$, we set

$$\mathcal{S}_i = \exp_{q_i^{-1}}(q_i^{-1} E_i K_i^{-1})\, \exp_{q_i^{-1}}(-F_i)\, \exp_{q_i^{-1}}(q_i E_i K_i)\, q_i^{\frac{h_i(h_i+1)}{2}}$$

Thus, if $\mathcal{U}_\hbar(\mathfrak{g}) \circlearrowright V$ (where E_i, F_i act locally nilpotently) and $v_\mu \in V[\mu]$ $(\mu \in \mathfrak{h}^*)$, then

$$\mathcal{S}_i v = \sum_{\substack{a,b,c\geq 0\\ b-a-c=\mu(h_i)}} (-1)^b q_i^{b-ac}\, \underbrace{E_i^{(a)}}_{\frac{E_i^a}{[a]_i!}} F_i^{(b)} E_i^{(c)} \cdot v_\mu \in V[s_i(\mu)].$$

Combining the calculations of §1 and Lemma 3, Cor. 4 - we obtain algebra automorphisms $T_i \circlearrowright \mathcal{U}_\hbar(\mathfrak{g})$ $(i \in I)$:

$$T_i(E_i) = -F_i K_i\ ;\quad T_i(F_i) = -K_i^{-1} E_i\ ;\quad T_i(h) = s_i(h)\ ;$$

$$(j \neq i) \quad T_i(E_j) = \sum_{s=0}^{-a_{ij}} (-1)^s q_i^{-s} E_i^{(-a_{ij}-s)} E_j E_i^{(s)},$$

$$T_i(F_j) = \sum_{s=0}^{-a_{ij}} (-1)^s q_i^{s} F_i^{(s)} F_j F_i^{(-a_{ij}-s)}.$$