Lecture 28

0. Summary of $\mathcal{U}_\hbar(\mathfrak{g})$. ($\mathfrak{g}$ = simple Lie algebra assoc. to Cartan matrix A, and symmetrizing integers $(d_i)_{i\in I}$)

- Presentation - generators $h\in\mathfrak{h}$, $\{E_i, F_i\}_{i\in I}$.

relations - $[h,h']=0\ \forall h,h'\in\mathfrak{h}$;

$[h,E_i] = \alpha_i(h)E_i$; $[h,F_i] = -\alpha_i(h)F_i\quad \forall h\in\mathfrak{h}, i\in I$.

$$[E_i,F_j] = \delta_{ij}\frac{K_i-K_i^{-1}}{q_i-q_i^{-1}} \qquad \left(K_i = q_i^{h_i}\ ;\ q_i = q^{d_i}\right)$$

For $i\neq j$: $\displaystyle\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix}1-a_{ij}\\ s\end{bmatrix}_{q_i} E_i^{1-a_{ij}-s}E_jE_i^s = 0$ (= same expression with F's).

- ~~Atom~~ Hopf algebra structure -

$\Delta(h) = h\otimes 1 + 1\otimes h$; $\Delta(E_i) = E_i\otimes 1 + K_i\otimes E_i$ (coproduct)

$\Delta(F_i) = F_i\otimes K_i^{-1} + 1\otimes F_i$

$S(h) = -h$; $S(E_i) = -K_i^{-1}E_i$; $S(F_i) = -F_iK_i$ (antipode)

$\varepsilon(h) = \varepsilon(E_i) = \varepsilon(F_i) = 0$ (counit)

- R-matrix = canonical tensor of the <u>non-degenerate Hopf pairing</u>.

$\mathcal{U}_\hbar^{\leq 0}\times\mathcal{U}_\hbar^{\geq 0}\to\mathbb{C}((\hbar))$ -
- $(1,x) = \varepsilon(x)$; $(y,1) = \varepsilon(y)$
- $(y, x_1x_2) = (\Delta(y), x_1\otimes x_2)$
  $(y_1y_2, x) = (y_1\otimes y_2, \Delta^{op}(x))$
- $(h,h') = \dfrac{(h,h')_0}{\ln(q)}$
  $(F_i,E_j) = \dfrac{\delta_{ij}}{q_i-q_i^{-1}}$.

$\to$ $(\mathcal{U}_\hbar(\mathfrak{g}), R)$ - is a quasi-triangular Hopf algebra.

1. Braid group action. - For each $i \in I$, we have an algebra auto.

$T_i : \mathcal{U}_\hbar(\mathfrak{g}) \to \mathcal{U}_\hbar(\mathfrak{g})$ given by : $\quad T_i(h) = s_i(h) (= h - \alpha_i(h) h_i)$.

$$T_i(E_i) = -F_i K_i \; ; \; T_i(F_i) = -K_i^{-1} E_i$$

$$(j \neq i) \qquad T_i(E_j) = \sum_{s=0}^{-a_{ij}} (-1)^s q_i^{-s} E_i^{(-a_{ij}-s)} E_j E_i^{(s)} \qquad \left(E_i^{(\ell)} = \frac{E_i^\ell}{[\ell]_{q_i}!} \right)$$

$$T_i(F_j) = \sum_{s=0}^{-a_{ij}} (-1)^s q_i^{s} F_i^{(s)} F_j F_i^{(-a_{ij}-s)} .$$

Remark. - T_i is "almost inner" - i.e. $T_i(x) = \mathcal{S}_i \, x \, \mathcal{S}_i^{-1}$ for

$$\text{(Lusztig)} \qquad \mathcal{S}_i = \exp_{q_i^{-1}}(q_i^{-1} E_i K_i^{-1}) \exp_{q_i^{-1}}(-F_i) \exp_{q_i^{-1}}(q_i E_i K_i) \, q_i^{\frac{h_i(h_i+1)}{2}}$$

(also studied / introduced by : Kirillov-Reshetikhin
Levendorskii - Soibelman)

$\{T_i\}_{i \in I}$ (and also $\{\mathcal{S}_i\}_{i \in I}$) satisfy <u>braid relations</u>. Recall that we have $m_{ij} \in \{0, 3, 4, 6\}$ according to $a_{ij} a_{ji} = 0, 1, 2$ or 3.

Braid rel^{n}s : $\quad \underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ terms}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ terms}}$

2. Poincaré - Birkhoff - Witt Theorem.

Let $\mathfrak{a}$ be a Lie algebra over $\mathbb{C}$, $\{x_j\}_{j\in J}$ a basis of $\mathfrak{a}$.

Given a total ordering on $\{x_j\}_{j\in J}$, PBW theorem states that <u>the set of ordered monomials in x_j's is a basis of $\mathcal{U}(\mathfrak{a})$</u>.

When $\mathfrak{a} = \mathfrak{g}$ = simple Lie algebra coming from a root system -

a basis of $\mathfrak{g}$: $\{h_i\}_{i\in I}$ (basis of $\mathfrak{h}$) ; e_α, f_α ($\alpha \in R_+$ - set of positive roots).

We use $B_W \circlearrowright U_\hbar(\mathfrak{g})$ to construct analogues of e_α, f_α's for $\alpha \in R_+$ not necessarily a simple root.

(e.g. if $\alpha = w(\alpha_i)$, $w = s_{j_1} \cdots s_{j_r}$ (reduced exp.), define

$$E_\alpha = T_{j_1} \cdots T_{j_r}(E_i) \;, \quad F_\alpha = T_{j_1} \cdots T_{j_r}(F_i) \;)$$

(depends on w and i but not on the red. exp. of w)

PBW Theorem for $U_\hbar(\mathfrak{g})$ was obtained by Rosso (1988) - $\mathfrak{g} = \mathfrak{sl}_n$. and Lusztig (1990) .. arbitrary $\mathfrak{g}$.

→ Ordered monomials in $\{E_\alpha, F_\alpha, h_i\}_{i\in I, \alpha \in R_+}$ form a (topological) basis (over $\mathbb{C}[[\hbar]]$) of $U_\hbar(\mathfrak{g})$.

Thus $U_\hbar(\mathfrak{g}) \cong U(\mathfrak{g})[[\hbar]]$ as $\mathbb{C}[[\hbar]]$ - module.

3. Another application of $B_W \circlearrowright U_\hbar(\mathfrak{g})$ - Kirillov-Reshetikhin formula for R.

(q-Weyl gp. & a mult. formula for the R-matrix).

A nice way to enumerate (and hence order) positive roots is as follows - (called normal ordering):

Let $w_0 \in W$ be the longest element (so. $\ell(w_0) = \# R_+$.)

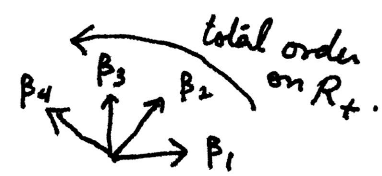
Choose a reduced expression $w_0 = s_{i_1} \cdots s_{i_\ell}$ and define

$$\beta_1 = \alpha_{i_1} \; ; \quad \beta_2 = s_{i_1}(\alpha_{i_2}) \; ; \; \dots \quad \beta_r = s_{i_1} \cdots s_{i_{r-1}}(\alpha_{i_r}) \; \dots$$

Since $s_{i_1} \cdots s_{i_\ell}$ is reduced ; $\beta_1, \dots, \beta_\ell$ are distinct positive roots and hence $R_+ = \{\beta_1, \dots, \beta_\ell\}$.
$(\ell = |R_+|)$

e.g. $w_0 = s_1 s_2 s_1$ (type A_2) $\rightsquigarrow$ $\beta_1 = \alpha_1$; $\beta_2 = s_1(\alpha_2) = \alpha_1 + \alpha_2$; $\beta_3 = s_1 s_2(\alpha_1) = s_1(\alpha_1 + \alpha_2) = \alpha_2$.

$w_0 = s_1 s_2 s_1 s_2$ (B_2) $\rightsquigarrow$ $\beta_1 = \alpha_1$; $\beta_2 = 2\alpha_1 + \alpha_2$; $\beta_3 = \alpha_1 + \alpha_2$; $\beta_4 = \alpha_2$



Kirillov-Reshetikhin formula :

$$\boxed{\mathcal{R} = q^{\Omega_0} \cdot \overset{\curvearrowright}{\prod_{\alpha \in R_+}} \exp_{q_\alpha}\left((q_\alpha - q_\alpha^{-1}) F_\alpha \otimes E_\alpha\right)}$$

where • $\Omega_0 \in \mathfrak{h} \otimes \mathfrak{h}$ is the canonical tensor of $(h_i, h_j)_0 = \frac{1}{d_j} a_{ij}$.

- product is ordered by $\beta_1 < \beta_2 < \cdots < \beta_\ell$.
- $E_{\beta_r} = T_{i_1} \cdots T_{i_{r-1}}(E_{i_r})$; $F_{\beta_r} = T_{i_1} \cdots T_{i_{r-1}}(F_{i_r})$ $\forall r \in \{1, \ldots, \ell\}$.

(Though E_α's, F_α's obtained as above depend on the reduced exp. of w_0 and also the ordering of positive roots ; the product written above does not.)

4. Kirillov-Reshetikhin obtained this explicit expression using the coproduct identity for Lusztig element $\mathbb{S}$:

$$\Delta(\mathbb{S}) = \mathbb{S} \otimes \mathbb{S} \cdot \boxed{\exp_q\left((q - q^{-1}) F \otimes E\right)}$$

$$\overset{!!}{\overline{\mathcal{R}}} = q^{\frac{-H \otimes H}{2}} \mathcal{R}.$$

This identity can be proved as follows:

- Check that both sides have the same commutation with $\Delta(E)$, $\Delta(F)$, $\Delta(H)$
- Compute the action of both sides on a cyclic vector $\xi \in L_m \otimes L_n$ (eg. $\xi = v_m \otimes v_0$. lowest $\otimes$ highest.)

$\rightarrow$ appeal to a separation of points argument.

5. Finite dimensional representations of $\mathcal{U}_\hbar(\mathfrak{g})$.

Let V be a finite-dim'l vector space over $\mathbb{C}$. We consider $\mathbb{C}[[\hbar]]$-linear action of $\mathcal{U}_\hbar(\mathfrak{g})$ on $V[[\hbar]]$ ($= V \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$) and refer to such representations as "finite-dimensional".

With this caveat in mind - representation theory of $\mathcal{U}_\hbar(\mathfrak{g})$ is entirely analoguous to that of $\mathcal{U}(\mathfrak{g})$. As we will see later $\mathcal{U}_\hbar(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{g})[[\hbar]]$ as algebras - which is responsible for this analogy.

For example. Irred. f.d. repns $\longleftrightarrow$ $P_+ = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0}\}$

$$\begin{array}{ccc} \text{Irred. f.d. repns} & \longleftrightarrow & P_+ \\ \cup & & \cup \\ L_\lambda & \longleftrightarrow & \lambda \end{array}$$

L_λ admits the following presentation :

- L_λ is generated (as a module over $\mathcal{U}_\hbar(\mathfrak{g})$) by a (non-zero) vector v subject to:

$$h_i v = \lambda(h_i) v \quad ; \quad E_i v = 0 \quad ; \quad F_i^{\lambda(h_i)+1} v = 0.$$
$$(\forall i \in I)$$

6. Verma Modules. Let $\lambda \in \mathfrak{h}^*$ and let M_λ be the (infinite-dim'l) representation of $\mathcal{U}_\hbar(\mathfrak{g})$ generated by a non-zero vector - denoted here by $\mathbb{1}_\lambda$ - subject to

$$h \cdot \mathbb{1}_\lambda = \lambda(h) \mathbb{1}_\lambda \quad ; \quad E_i \mathbb{1}_\lambda = 0 \quad \forall i \in I.$$

Alternately, $M_\lambda \cong \mathcal{U}_\hbar(\mathfrak{g}) \Big/ \text{Left-ideal generated by } \left\{ \begin{matrix} h - \lambda(h) & (h \in \mathfrak{h}) \\ E_i & (i \in I) \end{matrix} \right\}$

- Universal property: for any repn. $\mathcal{U}_\hbar(\mathfrak{g}) \circlearrowright V$,

$$\operatorname{Hom}_{\mathcal{U}_\hbar(\mathfrak{g})}(M_\lambda, V) = V[\lambda]^{\mathcal{U}^+} \quad \left(\begin{matrix}\text{singular vectors of} \\ \text{weight } \lambda\end{matrix}\right)$$
$$\| $$
$$\left\{ v \in V : \begin{matrix} h \cdot v = \lambda(h) v \ \forall h \in \mathfrak{h} \\ E_i \cdot v = 0 \ \forall i \in I \end{matrix} \right\}$$

- $$M_\lambda = \bigoplus_{\mu \in Q_+} M_\lambda[\lambda - \mu]$$

$\dim M_\lambda[\lambda-\mu] =$ Kostant's partition fn. (μ)

$= \#$ of ways of writing μ as sum of +ve roots.