Lecture 29 - Dynamical constructions*

0. For this part, we may assume that $q \in \mathbb{C}^\times$ $(q = e^{\hbar/2})$ is a "generic" complex number and denote the quantum group by $\mathcal{U}_q(\mathfrak{g})$.
[not a root of unity]

Recall: for $\lambda \in \mathfrak{h}^*$, we defined M_λ (Verma module) as the repn. generated by one cyclic (highest weight) vector $\mathbb{1}_\lambda$:

$$h \cdot \mathbb{1}_\lambda = \lambda(h)\,\mathbb{1}_\lambda \quad ; \quad E_i \cdot \mathbb{1}_\lambda = 0 \quad \forall i \in I.$$

$$\left(\text{or } K_i \cdot \mathbb{1}_\lambda = q_i^{\lambda(h_i)}\,\mathbb{1}_\lambda \ \forall i \in I\right)$$

1. Let $\mathcal{U}_q(\mathfrak{g}) \circlearrowright V$ be a finite-dimensional representation.

As before, $V[\mu] := \{ v \in V \mid h.v = \mu(h).v \ \forall h \in \mathfrak{h} \}$ ($\mu \in \mathfrak{h}^*$) (μ-weight space of V).

$P(V) := \{ \mu \in \mathfrak{h}^* \mid V[\mu] \neq 0 \}$ set of weights of V.

Prop. Let $\mu \in P(V)$, $\lambda \in \mathfrak{h}^*$. Then for "generic λ" - the map

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{U}_q(\mathfrak{g})}(M_\lambda, M_{\lambda-\mu} \otimes V) & \longrightarrow & V[\mu] \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \varphi & \longmapsto & \text{coeff of } \mathbb{1}_{\lambda-\mu} \text{ in } \varphi(\mathbb{1}_\lambda) \end{array}$$

is an iso. of vector spaces.

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* Etingof, Latour - Dynamical YBE, repn. theory & quantum integrable systems.

This proposition allows us to view weight vectors in V as $U_q(\mathfrak{g})$-intertwiners - depending on $\lambda \in \mathfrak{h}^*$.

$$v \in V[\mu] \rightsquigarrow \varphi_\lambda^v : M_\lambda \longrightarrow M_{\lambda-\mu} \otimes V$$

$$\mathbb{1}_\lambda \longmapsto \mathbb{1}_{\lambda-\mu} \otimes v + \cdots$$

terms from $M_{\lambda-\mu}[\lambda-\mu-\gamma] \otimes V[\mu+\gamma]$

$\gamma \in Q_+ \setminus \{0\}$.

eg. $\mathfrak{g} = sl_2$. $m \in \mathbb{Z}_{\geq 0}$; $\lambda \in \mathbb{C}$.

$V = L_m$: $(m+1)$-dim'l irred. with basis $\{v_0, \ldots, v_m\}$

M_λ has the following basis $\{F^{(\ell)} \cdot \mathbb{1}_\lambda\}_{\ell \geq 0}$ $\left(F^{(\ell)} = \dfrac{F^\ell}{[\ell]!}\right)$

Let $0 \leq k \leq m$. Then $\varphi_\lambda^{v_k} : M_\lambda \longrightarrow M_{\lambda-(m-2k)} \otimes V$

$$\mathbb{1}_\lambda \longmapsto \mathbb{1}_\gamma \otimes v_k + \sum_{\ell \geq 1} c_\ell \, (F^{(\ell)} \cdot \mathbb{1}_\gamma) \otimes v_{k-\ell}$$

$(\gamma = \lambda - (m-2k))$

coefficients to be determined.

The coefficients $\{c_\ell\}_{\ell=1}^{k}$ are uniquely determined by

$$E \cdot \left(\mathbb{1}_\gamma \otimes v_k + \sum_{\ell \geq 1} c_\ell \, (F^{(\ell)} \mathbb{1}_\lambda) \otimes v_{k-\ell}\right) = 0.$$

2. Fusion Operator. Let V_1, V_2 be two f.d. repns. of $U_q(\mathfrak{g})$

$v_1 \in V_1[\mu_1]$, $v_2 \in V_2[\mu_2]$. Consider the composition

$$M_\lambda \xrightarrow{\varphi_\lambda^{v_2}} M_{\lambda-\mu_2} \otimes V_2 \xrightarrow{\varphi_{\lambda-\mu_2}^{v_1} \otimes \mathrm{id}_{V_2}} M_{\lambda-\mu_1-\mu_2} \otimes V_1 \otimes V_2$$

<u>Definition</u>. The coefficient of $\mathbb{1}_{\lambda-\mu_1-\mu_2}$ in $(\varphi_{\lambda-\mu_2}^{v_1} \otimes \mathrm{id}_{V_2})(\varphi_\lambda^{v_2}(\mathbb{1}_\lambda))$

is denoted by $\mathcal{J}_{V_1,V_2}(\lambda)\cdot(v_1 \otimes v_2)$

$$\in (V_1 \otimes V_2)[\mu_1+\mu_2]$$

We view $\mathcal{J}_{V_1,V_2}(\lambda)$ as an $\mathrm{End}(V_1 \otimes V_2)$-valued fn. of λ.

Properties : (1) $J_{V_1,V_2}(\lambda)$ is zero weight, lower triangular with 1's on the diagonal

(2) $J_{V_1,V_2}(\lambda)$ is a rational function of q^λ.

Meaning - $\forall\ v_1 \in V_1[\mu_1]$, $v_2 \in V_2[\mu_2]$ -

$$J_{V_1,V_2}(\lambda)\cdot v_1 \otimes v_2 = v_1 \otimes v_2 + \text{terms from } V_1[\mu_1-\gamma] \otimes V_2[\mu_2+\gamma]$$

$(\gamma \in Q_+ \setminus \{0\})$

Coefficients are rational fns. of q^λ.

e.g. $V_1 = V_2 = \mathbb{C}^2$; $v_1 = \uparrow$ (weight $= +1$) ; $v_2 = \downarrow$ (weight $= -1$)

$$J_{\mathbb{C}^2,\mathbb{C}^2}(\lambda).\ \uparrow\downarrow \;=\; \uparrow\downarrow \;-\; \frac{q^{\lambda+1}}{[\lambda+1]}\downarrow\uparrow$$

$$\frac{q-q^{-1}}{1-q^{-2(\lambda+1)}}$$

3. Twist (or cocycle) equation.

Notation. - for $F : \mathfrak{h}^* \longrightarrow \mathrm{End}_{\mathbb{C}}(V_1 \otimes \ldots \otimes V_N)$; $F(\lambda + wt_j)$ denotes the function

$$F(\lambda + wt_j)\cdot(v_1 \otimes \ldots \otimes v_N) = F(\lambda+\mu_j)(v_1 \otimes \ldots \otimes v_N)$$

if $v_j \in V_j[\mu_j]$.

Prop.
$$J_{V_1\otimes V_2,\,V_3}(\lambda)\; J_{V_1,V_2}(\lambda - wt_3)$$
$$= J_{V_1,\,V_2\otimes V_3}(\lambda)\cdot J_{V_2,V_3}(\lambda)$$

Proof. Evaluated on $v_1 \otimes v_2 \otimes v_3$; where $v_j \in V_j[\mu_j]$; $j = 1,2,3$. both sides are coefficient of $\mathbb{1}_{\lambda-\mu_1-\mu_2-\mu_3}$ in the following composition of intertwiners - acting on $\mathbb{1}_\lambda$:

$$\begin{tikzcd} M_\lambda \arrow[r, "\varphi_\lambda^{v_3}"'] & M_{\lambda-\mu_3} \otimes V_3 \arrow[r, "\varphi_{\lambda-\mu_3}^{v_2} \otimes id"'] & M_{\lambda-\mu_2-\mu_3} \otimes V_2 \otimes V_3 \arrow[d, "\varphi_{\lambda-\mu_2-\mu_3}^{v_1} \otimes id \otimes id"] \\ & & M_{\lambda-\mu_1-\mu_2-\mu_3} \otimes V_1 \otimes V_2 \otimes V_3 \end{tikzcd}$$

$\square$

4. Singular vectors in M_λ.

Recall the commutation relation for $\mathcal{U}_q(\mathfrak{sl}_2)$:

$$[E, F^{(n)}] = \frac{q^{n-1} K - q^{-n+1} K^{-1}}{q - q^{-1}} \cdot F^{(n-1)}$$

This implies $\quad E \cdot (F^{(n)} \mathbb{1}_\lambda) = [\lambda - n + 1] \, (F^{(n-1)} \mathbb{1}_\lambda)$

$= 0 \quad$ if $\quad n = \lambda + 1 \quad$ (i.e. $\lambda \in \mathbb{Z}_{\geq 0}$ here).

In general : $\quad \lambda \in \mathfrak{h}^*, \quad \lambda(h_i) \in \mathbb{Z}_{\geq 0}$ (for an $i \in I$) $\Rightarrow F_i^{\lambda(h_i)+1} \cdot \mathbb{1}_\lambda$ is a singular vector - in the weight space $\lambda - (\lambda(h_i) + 1)\alpha_i$

$= s_i(\lambda) - \alpha_i$

Shifted action of W on $\mathfrak{h}^*$: let $\rho \in \mathfrak{h}^*$ be such that $\rho(h_i) = 1 \;\forall i$

Then $\quad s_i(\lambda) - \alpha_i = s_i(\lambda + \rho) - \rho =: s_i \cdot \lambda$

$$w \cdot \lambda := w(\lambda+\rho) - \rho .$$

Verma Identities - given $w \in W$, let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression. Set $\beta_1 = \alpha_{i_1}$; $\beta_2 = s_{i_1}(\alpha_{i_2})$; $\cdots$; $\beta_\ell = s_{i_1} \cdots s_{i_{\ell-1}}(\alpha_{i_\ell})$.

$$n_j = \frac{2}{(\beta_j, \beta_j)} (\lambda+\rho, \beta_j)$$ (assume this is positive integer.)

$\rightsquigarrow$ $f_{i_\ell}^{(n_\ell)} \cdots f_{i_1}^{(n_1)} \cdot \mathbb{1}_\lambda \in M_\lambda[w \cdot \lambda]$ is a singular vector independent of the choice of the reduced expression.

5. Dynamical Weyl group. - sl_2 case. (assume $\lambda, \lambda-\mu \in \mathbb{Z}_{>0}$).

$U_q(sl_2) \circlearrowright V$; $v \in V[\mu]$. $\varphi_\lambda^v : M_\lambda \longrightarrow M_{\lambda-\mu} \otimes V$

f.d. repn.

Lemma. $\varphi_\lambda^v \left(F^{(\lambda+1)} \mathbb{1}_\lambda \right) = F^{(\lambda-\mu+1)} \cdot \mathbb{1}_{\lambda-\mu} \otimes \tilde{v} + \cdots$

for λ sufficiently large compared to v. - so that $V[\mu - 2\lambda - 2] = 0$.

This $\tilde{v} \in V[-\mu]$ is defined to be $\boxed{A_{s;V}(\lambda) \cdot v}$.

$$\begin{tikzcd}
F^{(\lambda+1)}\mathbb{1}_\lambda & M_\lambda \arrow[rr, "\varphi_v^\lambda"] & & M_{\lambda-\mu}\otimes V \\
\mathbb{1}_{-\lambda-2} \arrow[u, mapsto] \arrow[r, phantom, "\in"] & M_{-\lambda-2} \arrow[u, hook] \arrow[rr] & & M_{-\lambda+\mu-2}\otimes V \arrow[u, hook] \\
\mathbb{1}_{-\lambda-2} \arrow[rr, mapsto, "(\text{restriction of } \varphi_v^\lambda)"'] & & \mathbb{1}_{-\lambda+\mu-2}\otimes \boxed{A_{s;V}(\lambda).v} + \cdots &
\end{tikzcd}$$

Picture defining $A_{s;V}(\lambda) : V \to V$.

6. Dynamical Weyl group – general case:

$w = s_{i_1}\cdots s_{i_\ell}$ reduced expression $\rightsquigarrow$ define

shifted action

$$A_{w;V}(\lambda) = A_{s_{i_1}}(s_{i_2}\cdots s_{i_\ell}\cdot\lambda)\ \cdots\ A_{s_{i_{\ell-1}};V}(s_{i_\ell}\cdot\lambda)\ A_{s_{i_\ell};V}(\lambda)$$

Induction argument & Verma identities imply that $A_{w;V}(\lambda).v$ $(v\in V[\mu])$

is the coefficient of $\mathbb{1}_{w\cdot(\lambda-\mu)}$ in $\varphi_\lambda^v\big|_{M_{w\cdot\lambda}}$ – hence independent of the choice of the reduced expression.

7. Coproduct Identity

$$\boxed{\begin{aligned} &A_{w;V_1\otimes V_2}(\lambda)\,J_{V_1,V_2}(\lambda) \\ &\quad = J_{V_1,V_2}(w\cdot\lambda)\cdot A_{w;V_1}(\lambda-\mathrm{wt}_2)\otimes A_{w;V_2}(\lambda)\end{aligned}}$$

Proof. Both sides $\overset{\text{on } V_1 \otimes V_2}{\text{give}}$ the coeff of $\mathbb{1}_{\omega\cdot(\lambda-\mu_1-\mu_2)}$ $\left(\begin{smallmatrix} v_1 \in V_1[\mu_1] \\ v_2 \in V_2[\mu_2]\end{smallmatrix}\right)$

in $\varphi^{v_1}_{\lambda-\mu_2} \otimes \mathrm{id} \circ \varphi^{v_2}_{\lambda} \Big|_{M_{\omega\cdot\lambda}}$ - see below:

$$\begin{tikzcd}
M_\lambda \arrow[r] & M_{\lambda-\mu_2} \otimes V_2 \arrow[r] & M_{\lambda-\mu_1-\mu_2} \otimes V_1 \otimes V_2 \\
M_{\omega\cdot\lambda} \arrow[u, hook] \arrow[r, dashed] & M_{\omega\cdot(\lambda-\mu_2)} \otimes V_2 \arrow[u, dashed, leftrightarrow] \arrow[r, dashed] & M_{\omega\cdot(\lambda-\mu_1-\mu_2)} \otimes V_1 \otimes V_2 \arrow[u, hook]
\end{tikzcd}$$

$\square$

8. Limits as $q^\lambda \to 0$ or ∞. All the operators we considered are rational fns. of q^λ. regular at $q^\lambda = 0$ or ∞.

$J(\lambda) \longrightarrow (\bar{R})^{-1}$ at $q^\lambda = \infty$ (see example on page 4)

$\longrightarrow 1$ at $q^\lambda = 0$

$A_{s;V}(\lambda) \longrightarrow (\pm 1)\cdot \underset{\substack{\uparrow \\ \text{Lusztig element}}}{\mathbb{S}}$ at $q^\lambda = \infty$

$\Rightarrow$ Braid relations for Lusztig operators - from §6.

Coproduct id. $\Delta(\mathbb{S}) = \mathbb{S} \otimes \mathbb{S} \cdot \bar{R}$ - from §7.