Lecture 3

ODE's over $\mathbb{C}$: irregular case.

0. We will now consider the differential equation $F'(z) = A(z)F(z)$ when $A(z)$ has a pole of order >1 at $z=0$. Recall that if the order of this pole is $r+1$, we say that $0 \in \mathbb{C}$ is an irregular singularity of Poincaré rank r.

For our purposes, it will be enough to consider the case $r=1$.

That is,
$$A(z) = \frac{\Lambda}{z^2} + \frac{X}{z} + \underbrace{\sum_{k=0}^{\infty} A_k z^k}_{A_{reg}(z)}$$

We will further assume that $\Lambda \in M_{N\times N}(\mathbb{C})$ is a diagonal matrix with distinct eigenvalues :

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix} \quad ; \quad \lambda_i \neq \lambda_j \quad \forall\, i\neq j \in \{1,\dots,N\}.$$

1. Formal solution. Let $X = (x_{ij})_{1\le i,j\le N} \in M_{N\times N}(\mathbb{C})$.

We use the following notation $X^d = \begin{bmatrix} x_{11} & & 0 \\ & \ddots & \\ 0 & & x_{NN} \end{bmatrix}$ (diagonal part of X)

$$X^{o\cdot d} = X - X^d.$$

Theorem. There exists a unique formal series

$$Y(z) = 1 + Y_1 z + Y_2 z^2 + \dots$$

such that $\psi(z) = Y(z) . z^{X^d} . e^{-\Lambda/z}$ is a solution of $F'(z) = A(z) F(z)$.

Proof. $\psi(z) = Y(z) . z^{X^d} . e^{-\Lambda/z}$ solves the required equation

if, and only if
$$Y'(z) = z^{-2}[\Lambda, Y(z)] + z^{-1}[X^d, Y(z)] + z^{-1} X^{o.d.} Y(z) + A_{reg}(z) . Y(z).$$

Coefficient of z^{-2} : $\quad 0 = [\Lambda, Y_0] = [\Lambda, \mathrm{Id}]$ ✓

Coefficient of z^{-1} : $\quad 0 = [\Lambda, Y_1] + \underbrace{\cancel{[X^d, Y_0]}}_{0 \text{ (since } Y_0 = 1)} + X^{o.d.}$

determines $Y_1^{o.d.}$ as : $\quad (Y_1)_{ij} = \dfrac{-1}{\lambda_i - \lambda_j} X_{ij} \quad (\forall i \neq j)$.

Coefficient of z^0 : $\quad Y_1 = [\Lambda, Y_2] + [X^d, Y_1] + X^{o.d.} Y_1 + A_0 . Y_0$

This equation, simultaneously, determines Y_1^d (and hence Y_1) and $Y_2^{o.d.}$:

- $$(Y_1)_{ii} = (X^{o.d} . Y_1)_{ii} + (A_0)_{ii} = (A_0)_{ii} + \sum_{\substack{j=1 \\ (j \neq i)}}^{N} x_{ij} (Y_1)_{ji}$$

- for $i \neq j$, $$(\lambda_i - \lambda_j)(Y_2)_{ij} = (1 - x_{ii} + x_{jj})(Y_1)_{ij} - (X^{o.d.} Y_1 + A_0)_{ij}$$

Continuing in this fashion, we conclude that, $\forall\, n \geq 1$, comparing coefficients of z^n, gives:

$$(n+1)\,Y_{n+1} = [\Lambda, Y_{n+2}] + [X^d, Y_{n+1}] + X^{o.d.}\,Y_{n+1} + \sum_{j=0}^{n} A_j Y_{n-j}\,.$$

By induction, we already have obtained $\{Y_0, \ldots, Y_n\}$ and $Y_{n+1}^{o.d.}$. Thus, this equation determines uniquely Y_{n+1}^{d} (& hence Y_{n+1}) and $Y_{n+2}^{o.d}$. $\square$

2. Example. $$\frac{dF}{dz} = \Bigg(z^{-2} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\Lambda} + z^{-1} \underbrace{\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}}_{X} \Bigg) F$$

(so $X = X^{o.d.}$, $X^d = 0$).

Equation for the formal series $Y(z)$ from Theorem above:

(write $Y(z) = \begin{bmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{bmatrix}$; $\alpha(0) = 1 = \delta(0)$, $\beta(0) = 0 = \gamma(0)$).

$$\cancel{dF} \quad \frac{dY}{dz} = z^{-2} \begin{bmatrix} 0 & \beta(z) \\ -\gamma(z) & 0 \end{bmatrix} + z^{-1} \begin{bmatrix} -\gamma(z) & -\delta(z) \\ 0 & 0 \end{bmatrix}.$$

This immediately implies that $\left.\begin{matrix} \gamma'(z) = -z^{-2}\gamma(z) \\ \gamma(0) = 0 \end{matrix}\right\} \Rightarrow \gamma \equiv 0.$

$$\alpha'(z) = -z^{-1}\gamma(z) = 0 \quad \Rightarrow \quad \alpha \equiv 1.$$
$$\alpha(0) = 1$$

$\delta'(z) = 0 \Rightarrow \delta \equiv 1$. For the (1,2) entry, we get

$$\beta'(z) = \frac{\beta(z)}{z^2} - \frac{\delta(z)}{z} = \frac{\beta(z)}{z^2} - \frac{1}{z}$$

$$\beta(z) = \sum_{n \geq 1} b_n z^n.$$

Solving for b_n, step-by-step in n, we get: $b_1 = 1$; $b_2 = 1$; $b_3 = 2b_2 = 2$; ... ; $(n+1)\, b_{n+1} = b_{n+2}$.
(i.e. $b_{n+2} = (n+1)!$)

Hence $\boxed{\beta(z) = \sum_{n \geq 1} z^n \cdot (n-1)!}$ has zero radius of convergence.

3. As the example above illustrates, the formal series obtained in Theorem 1. above does not converge (in general), and hence does not represent a holomorphic function, near 0. However, it can be viewed as an <u>asymptotic expansion</u> of <u>some</u> function (not unique) as $z \to 0$ in a prescribed manner.

<u>Definition</u>. (i) Given $R > 0$ and $-\pi < \theta_1 < \theta_2 < \pi$; let

$$S(R; \theta_1, \theta_2) := \{ z \in \mathbb{C} \mid |z| < R \text{ and } \arg(z) \in (\theta_1, \theta_2) \}$$

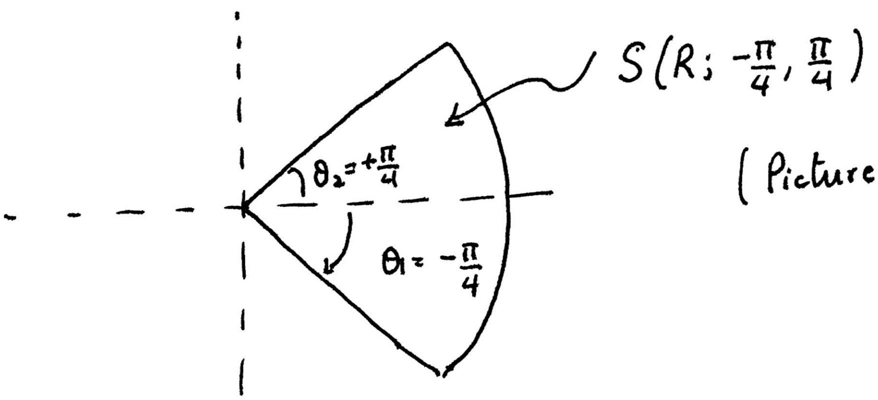
(open <u>sector</u> near 0).

(ii) Let $f(z)$ be a holomorphic function on $S(R;\theta_1,\theta_2)$ and $\sum_{k=0}^{\infty} a_k z^k \in \mathbb{C}[[z]]$. We say

$$f(z) \sim \sum_{k=0}^{\infty} a_k z^k \quad \text{as } z\to 0 \text{ in } S(R;\theta_1,\theta_2)$$

(read: $f(z)$ is asymptotic to $\sum_{k=0}^{\infty} a_k z^k$ as z approaches 0 within the sector $S(R;\theta_1,\theta_2)$.)

if, for every $M \geq 0$, $\displaystyle\lim_{\substack{z\to 0\\ z\in S(R;\theta_1,\theta_2)}} z^{-M}\left(f(z) - \sum_{k=0}^{M-1} a_k z^k\right)$ exists.



(Picture of a typical sector).

e.g. $e^{-\frac{1}{z}} \sim 0$ as $z \to 0$, $z\in S(1; -\frac{\pi}{4}, \frac{\pi}{4})$. This example shows that, unlike convergent series, an asymptotic series <u>does not</u> determine the function. In fact there are infinitely many functions (on a fixed sector) which have the same asymptotic expansion as $z\to 0$. For instance, $e^{-\frac{1}{z^b}}$ $(0<b<1)$ are all asymptotic to 0.

4. Some facts about asymptotic series

(Cf. W. Wasow: *Asymptotic expansions for ODE's*. Ch III. §8, 9.)

- Termwise addition, multiplication is valid for asymptotic series. (and composition)

- Term-wise integration: $f(x) \sim \sum_{r=0}^{\infty} a_r x^r$ (as $x \to 0$, $x \in S$)

$$\Rightarrow \int_0^x f(t)\,dt \sim \sum_{r=0}^{\infty} a_r \frac{x^{r+1}}{r+1}$$ provided the path of integration lies within S.

- Termwise differentiation. (only valid for sectors with non-zero opening angle).

$S = S(R;\theta_1, \theta_2)$ where $\theta_1 < \theta_2$

$$f(x) \sim \sum_{r=0}^{\infty} a_r x^r$$

$\Rightarrow \forall \cancel{\exists}\ \theta_1' < \theta_2'$; $\theta_1 < \theta_1' < \theta_2' < \theta_2$

$$f'(x) \sim \sum_{r=1}^{\infty} a_r \cdot r \cdot x^{r-1}$$ as $x \to 0$ in $S(R;\theta_1', \theta_2')$.

(Non-example: $f(x) = e^{-\frac{1}{x}} \cancel{\sin(1/x)} \sin(e^{1/x}) \sim 0$ as $\begin{pmatrix} x \to 0 \\ x \in \mathbb{R}_{>0} \end{pmatrix}$

$$f'(x) = x^{-2}\left(e^{-\frac{1}{x}} \sin(e^{1/x}) + \cos(e^{\frac{1}{x}})\right)$$

has <u>no</u> asymptotic expansion as $x \to 0$, $x \in \mathbb{R}_{>0}$.)

- (a theorem of J.F. Ritt (1914))

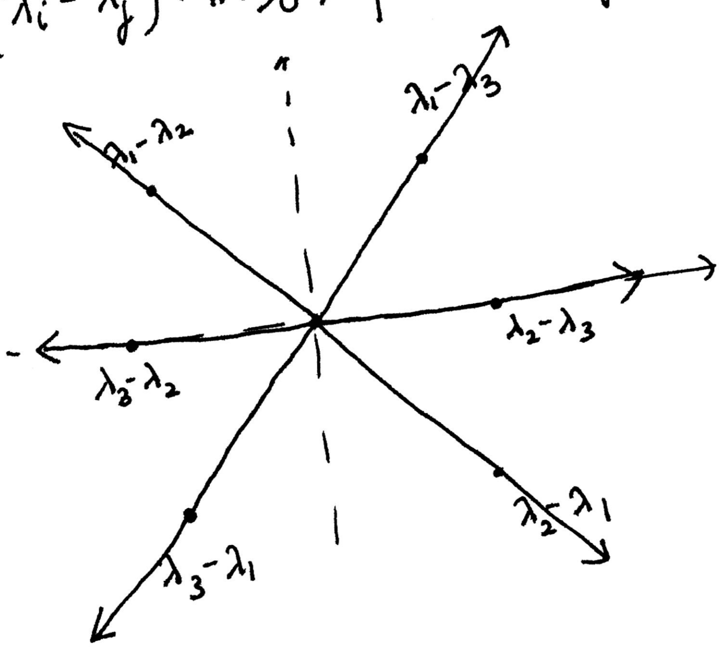
Given a sector $S(R;\theta_1,\theta_2)$ where $\theta_1<\theta_2$ and a formal series $\sum_{r=0}^{\infty} a_r x^r$; there exists a function $f(x)$, holomorphic on $S(R;\theta_1,\theta_2)$ such that $f(x)\sim\sum_{r=0}^{\infty} a_r x^r$ as $x\to 0$, $x\in S$.

5. Back to our case (see §0 above). Let us drop $A_{reg}(z)$ and assume that our differential equation only features two matrices Λ and X; recall: $\Lambda=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_N\end{bmatrix}$ $\lambda_i\neq\lambda_j$ for $i\neq j$.

(cf. P.P. Boalch - Stokes matrices, Poisson Lie groups and Frobenius manifolds. (Invent. Math 2001).)

Some definitions: An <u>anti-Stokes ray</u> is a ray in the complex plane of the form $(\lambda_i-\lambda_j)\cdot\mathbb{R}_{>0}$; for some $i\neq j$.

e.g. $\begin{bmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ 0 & & \lambda_3\end{bmatrix}\rightsquigarrow$

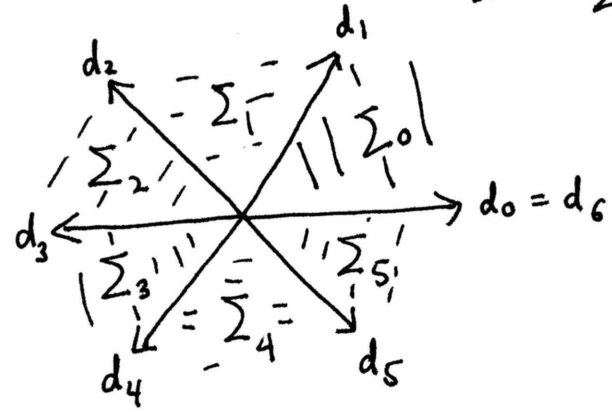


Six anti-Stokes rays.

Let $2 \cdot \ell$ be the number of (anti) Stokes rays. Let us order them in a counter clockwise fashion $(d_0, d_1, \ldots, d_{2\ell-1}, d_{2\ell} = d_0)$

Let Σ_i = Sector between d_i and d_{i+1}.

Picture of sectors Σ_i



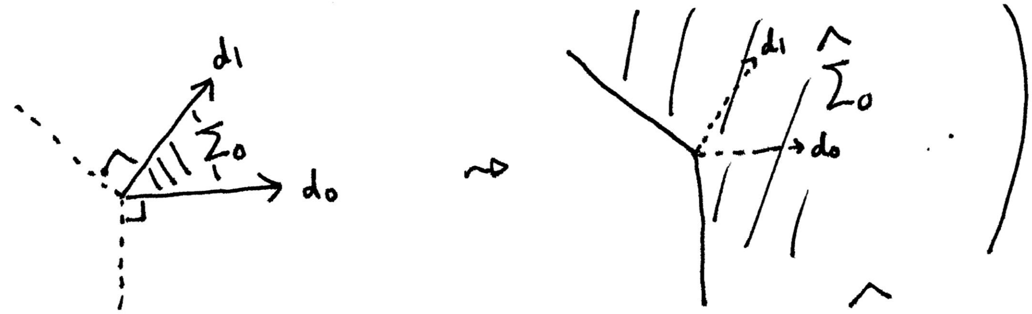
Theorem. (i) For every $i \in \{0, \ldots, 2\ell-1\}$ there exists a holomorphic function $Y_i(z)$ on Σ_i s.t.

$\psi_i(z) = Y_i(z) \cdot z^{X^d} e^{-\frac{\Lambda}{z}}$ solves

$$F'(z) = \left(\frac{\Lambda}{z^2} + \frac{X}{z}\right) F.$$

(ii) Let $\hat{\Sigma}_i$ be the larger sector between $d_i - \frac{\pi}{2}$ and $d_{i+1} + \frac{\pi}{2}$

(Fig:



)

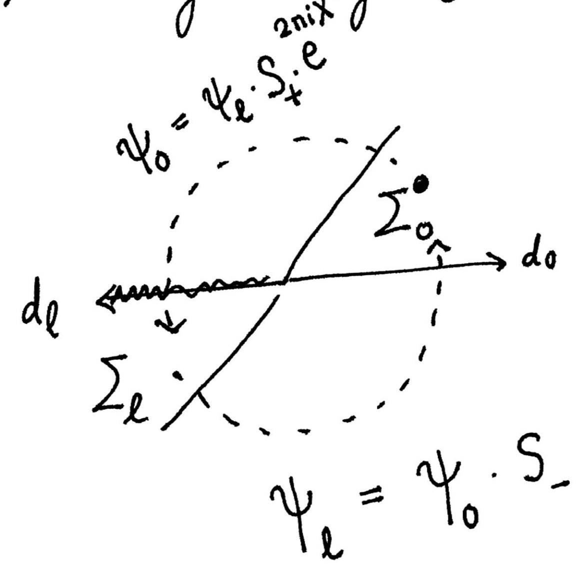
Then $Y_i(z) \sim Y(z)$ (from Thm 1) as $z \to 0$, $z \in \hat{\Sigma}_i$.

(rays bounding $\hat{\Sigma}_i$ are called Stokes rays.)

6. Stokes matrices (S_-, S_+) relate solutions $\psi_0(z)$ and $\psi_\ell(z)$.

- assume the cut for $\ln(z)$ is along the ray d_ℓ.

This is best depicted via: $\rightsquigarrow$



$$\begin{cases} \psi_\ell = \psi_0 \cdot S_- \\ \psi_0 = \psi_\ell \cdot S_+ \cdot e^{2\pi i X^0} \end{cases}$$

7. $$\boxed{\beta(z) = \sum_{n \geq 1} (n-1)!\, z^n}$$ from § 2

[Hardy: Divergent Series; § 2.4. – Euler and $\sum_{n=0}^{\infty} x^n \cdot n!$]

(Euler) – Let $\ell_\phi = r \cdot e^{i\phi}$ (ϕ fixed phase $\neq 0$. r goes from 0 to ∞)

$$\int_{\ell_\phi} \frac{e^{-p z^{-1}}}{1-p}\, dp \sim \sum_{n \geq 1} (n-1)!\, z^n \quad \text{in } S\left(R; \phi - \tfrac{\pi}{2}, \phi + \tfrac{\pi}{2}\right)$$

[Recommended: O. Costin – Asymptotics & Borel Summability pp. 108, § 4.4d.]