Lecture 30

0. The aim of the next few lectures is to prove the Kohno-Drinfeld theorem: let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Recall that we associated, to $\mathfrak{g}$, <u>two</u> (quasi) bialgebras:

- $(U(\mathfrak{g})[[\hbar]], \Delta_0, R_{KZ}, \Phi_{KZ})$. Here $\Delta_0(x) = x\otimes 1 + 1\otimes x \quad \forall x \in \mathfrak{g}$.

$$R_{KZ} = e^{\frac{\hbar}{2}\Omega} \quad \text{from KZ equation on 2 points}$$

$$\Phi_{KZ} = \text{associator defined by KZ equations on 3 points}$$

(see Lecture 21 - page 2)

- $(U_\hbar(\mathfrak{g}), \Delta, R, 1^{\otimes 3})$. Quantum group defined by $\mathfrak{g}$. - it is a quasi triangular bialgebra.

Theorem. – These two are "twist equivalent". In more detail: there exist (i) an algebra iso. $\psi : U_\hbar(\mathfrak{g}) \to U(\mathfrak{g})[[\hbar]]$; and

(ii) $J \in U(\mathfrak{g})\otimes U(\mathfrak{g})[[\hbar]]$ invertible s.t. $\varepsilon\otimes 1(J) = 1 = 1\otimes\varepsilon(J)$.
(twist)

so that the following equations hold:

$$\forall x \in U_\hbar(\mathfrak{g}) : \quad \psi\otimes\psi(\Delta(x)) = J\,\Delta_0(\psi(x))\,J^{-1}$$

$$\psi\otimes\psi(R) = J_{21}\cdot R_{KZ}\cdot J^{-1}$$

$$1^{\otimes 3} = 1\otimes J \cdot 1\otimes\Delta_0(J)\cdot \Phi_{KZ}\cdot \Delta_0\otimes 1(J)^{-1} \cdot J^{-1}\otimes 1$$

In other words – use ψ to transport the structure from $U_\hbar(\mathfrak{g})$ to $U(\mathfrak{g})[[\hbar]]$. Then J provides the twist relating

$$\left(U(\mathfrak{g})[[\hbar]],\ \Delta^{\psi} = \psi\otimes\psi\circ\Delta\circ\psi^{-1},\ \psi\otimes\psi(R),\ 1^{\otimes 3}\right) \text{ to}$$
$$\left(U(\mathfrak{g})[[\hbar]],\ \Delta_0,\ R_{KZ},\ \Phi_{KZ}\right)$$

The existence of ψ (and J_1 s.t. $\Delta^{\psi} = J_1 \cdot \Delta_0 \cdot J_1^{-1}$) is proved using deformation theory arguments.

1. Hochschild cohomology. – Let A be an associative algebra over $\mathbb{C}$, and M be an A-A bimodule.

<u>(Co) Chain Complex</u> <u>$C^\bullet(A;M)$</u> – is defined as:

$$C^k(A;M) := \mathrm{Hom}_{\mathbb{C}}(A^{\otimes k}, M) = \left\{ \begin{array}{l} \mathbb{C}\text{- multi-linear maps} \\ \underbrace{A\times\dots\times A}_{k\text{-times}} \to M. \end{array} \right\}$$

$$d^k : C^k(A;M) \longrightarrow C^{k+1}(A;M)$$
$$\xi \longmapsto d\xi(a_0,\dots,a_k)$$
$$= a_0\cdot\xi(a_1,\dots,a_k) + \sum_{i=0}^{k-1}(-1)^{i+1}\xi(\dots,\underbrace{a_i a_{i+1}}_{\text{multiplied together}},\dots)$$
$$+ (-1)^{k+1}\xi(a_0,\dots,a_{k-1})\cdot a_k$$

Easy check: $d^{k+1}\circ d^k = 0$.

$$HH^k(A;M) \quad := \quad \frac{\mathrm{Ker}(d^k)}{\mathrm{Im}(d^{k-1})} .$$

(k^{th} Hochschild Cohomology group of A, valued in M)

2. Deformations of A. - Again let A be an associative algebra over $\mathbb{C}$. By <u>deformation of A</u>, we mean an associative, $\mathbb{C}[\![\hbar]\!]$-bilinear multiplication $* : A[\![\hbar]\!] \times A[\![\hbar]\!] \to A[\![\hbar]\!]$ s.t. $\forall a, b \in A$; we have:

$$a * b = \underbrace{a \cdot b}_{\boxed{\text{original mult. of } A}} + \sum_{n \geq 1} \hbar^n \mu_n(a,b) .$$

Hence, a deformation is prescribed by a family of $\mathbb{C}$-bilinear maps $\mu_n : A \times A \to A$ $(n \geq 0)$ s.t

(i) $\mu_0(a,b) = ab \quad \forall a, b \in A$.

(ii) Associativity: $\forall a, b, c \in A$ and $n \in \mathbb{Z}_{\geq 0}$ - we have

$$\boxed{\sum_{k+l=n} \mu_k(\mu_l(a,b), c) = \sum_{i+j=n} \mu_i(a, \mu_j(b,c))} \quad - (2.1)$$

Let $*$ and $*'$ be two deformed products:

$$a * b = \sum_{n \geq 0} \hbar^n \mu_n(a,b) \quad ; \quad a *' b = \sum_{n \geq 0} \hbar^n \mu_n'(a,b) .$$

We say $*$ and $*'$ are equivalent - if $(A[\![\hbar]\!], *)$ and $(A[\![\hbar]\!], *')$ are isomorphic, as algebras, by an iso. that is Id_A modulo $\hbar$. That is, there exists $\varphi : A[\![\hbar]\!] \to A[\![\hbar]\!]$, $\mathbb{C}[\![\hbar]\!]$-linear, s.t.

(i) $\varphi(a) = a \pmod{\hbar} \quad \forall a \in A$.

(ii) $\varphi(a * b) = \varphi(a) *' \varphi(b). \quad \forall a, b \in A$.

Thus, if $\varphi(a) = \sum_{n \geq 0} \hbar^n \varphi_n(a)$, then $\varphi_0 = \mathrm{Id}_A$ and $\forall a, b \in A$ and $n \in \mathbb{Z}_{\geq 0}$:

$$\boxed{\sum_{k+l=n} \varphi_k(\mu_l(a,b)) = \sum_{i+j+k=n} \mu'_i(\varphi_j(a), \varphi_k(b))} \quad - (2.2)$$

3. Proposition.- $HH^2(A; A) = 0 \Rightarrow$ any deformation of A is equivalent to the <u>trivial one</u> (Trivial def. $\equiv a * b = ab \;\; \forall a, b \in A$)

Proof. - Let $*$ be a deformation of A. Thus we have $\{\mu_n : A \times A \to A\}_{n \geq 0}$ satisfying (2.1). We will construct recursively, deformations $* = *_1, *_2, *_3, \cdots$ of A s.t. $a *_k b = ab + \hbar^k(\cdots)$; and these are all equivalent.

<u>Initial Step</u>. Consider the associativity eq^n (2.1) above - for $n=1$:

$$\mu_1(a,b)\cdot c + \mu_1(ab,c) = a\cdot\mu_1(b,c) + \mu_1(a,bc)$$

$\Rightarrow$ $d^2\mu = 0$. If $\boxed{HH^2(A;A)=0}$ then $\exists\ f \in C^1(A;A)$ (= $\mathbb{C}$-linear map $A\to A$)

s.t. $\mu_1 = df$; i.e.,

$$\mu_1(a,b) = af(b) - f(ab) + f(a)b$$

So, take $\varphi : A[\![\hbar]\!] \to A[\![\hbar]\!]$; $\varphi(a) := a + \hbar f(a)$ and define $*_2$ by transporting $*_1$ via φ (hence an equivalent deformation).

<u>Check</u>: $a *_2 b = ab + \hbar^2(\cdots)$. That is, if

$$a *_2 b = \sum_{k\geq 0} \mu_k^{(2)}(a,b)\hbar^k, \text{ then } \mu_1^{(2)} \equiv 0.$$

Proof.
$$\begin{aligned}\varphi(a) *_2 \varphi(b) &= \varphi(a*b)\\ &= \varphi(ab + \hbar\mu_1(a,b) + \cdots)\\ &= ab + \hbar(f(ab) + \mu_1(a,b)) + \cdots\end{aligned}$$

and
$$\begin{aligned}\varphi(a) *_2 \varphi(b) &= (a + \hbar f(a)) *_2 (b + \hbar f(b))\\ &= ab + \hbar(f(a)b + af(b) + \mu_1^{(2)}(a,b)) + \cdots\end{aligned}$$

$$\Rightarrow \cancel{f(a)b} + \cancel{af(b)} + \mu_1^{(2)}(a,b) = \cancel{f(ab)} + \cancel{\mu_1(a,b)} \quad (\text{coeff. of } \hbar)$$

$$\Rightarrow \mu_1^{(2)}(a,b) = 0$$

$\square$

In general, assume we have constructed $*_N \sim *$ s.t.

$$a *_N b = ab + \hbar^N (\cdots)$$
$$= ab + \hbar^N \mu_N^{(N)}(a,b) + O(\hbar^{N+1})$$

Then associativity equation (2.1) for $*_N$ – $n = N$ – $\Rightarrow \mu_N^{(N)} \in Z^2(A;A)$

$\Rightarrow \exists\, f_N : A \to A$ s.t. $\mu_N^{(N)} = d f_N$. Let $\varphi : A[[\hbar]] \to A[[\hbar]]$ be $\varphi(a) = a + \hbar^N f_N(a)$ and $*_{N+1} = *_N$ transported via φ. Same proof as above gives $a *_{N+1} b = ab + \hbar^{N+1}(\ldots)$ and we are done. □

4. Prop. – If $HH^1(A;A) = 0$, then any algebra iso $A[[\hbar]] \xrightarrow{I} A[[\hbar]]$ s.t. $I = \mathrm{Id}_A + \hbar I_1 + \cdots$ is inner (that is, $\exists\, \tilde{a} \in A[[\hbar]]$ s.t. $\cancel{\varphi(x}$ $I(x) = \tilde{a}\, x\, \tilde{a}^{-1}$.).

($\tilde{a} = 1 + \ldots$ hence invertible)

Proof. – Let $n \geq 1$ be smallest s.t. $I_n \not\equiv 0$. Since $I(ab) = I(a)\,I(b)$, taking coefficient of $\hbar^n$ gives $I_n(ab) = a\,I_n(b) + I_n(a)\,b$

$\Rightarrow \quad dI_n = 0 \quad$ i.e. $I_n \in Z^1(A;A)$.

$HH^1(A;A) = 0 \;\Rightarrow\; I_n = d(a_n)$ for some $a_n \in C^0(A,A) \cong A$.

That is $\quad I_n(x) = a_n x - x a_n$.

Define $\tilde{I}(x) = (1 - \hbar^n a_n)\, I(x)\, (1 - \hbar^n a_n)^{-1}$
$= x + O(\hbar^{n+1})$. Prop. is proved by repeating this argument. □

5. When $A = \mathcal{U}(\mathfrak{g})$ - Hochschild cohomology is same as the cohomology of <u>Chevalley - Eilenberg complex</u> - defined below.

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g} \circlearrowright M$ be a representation.

$$C^n(\mathfrak{g};M) := \mathrm{Hom}_{\mathbb{C}}(\Lambda^n \mathfrak{g}, M)$$

(alternating multilinear maps $\mathfrak{g} \times \dots \times \mathfrak{g} \longrightarrow M$.)

$$C^{n-1}(\mathfrak{g};M) \xrightarrow{d} C^n(\mathfrak{g};M) \text{ is given by :}$$

$\omega \in C^{n-1}(\mathfrak{g};M)$

x_i is skipped

$$d\omega(x_1, \dots, x_n) = \sum_{i=1}^{n} (-1)^{i-1} x_i \cdot \omega(x_1 \dots \hat{x}_i \dots x_n) + \sum_{1 \le i < j \le n} (-1)^{i+j} \omega([x_i x_j], x_1 \dots \hat{x}_i \dots \hat{x}_j \dots x_n)$$

Check - $d \circ d = 0$.

$$\boxed{H^n(\mathfrak{g};M) = \frac{\mathrm{Ker}(d^n)}{\mathrm{Im}(d^{n-1})}} \text{ Lie alg. coh.}$$

6. The relation between the two cohomology theories is given as follows. Let M be a $\mathcal{U}(\mathfrak{g}) - \mathcal{U}(\mathfrak{g})$ bimodule

$\overline{M} = M$ as a vector space

$$\boxed{i : \mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})}$$

$\mathfrak{g} \circlearrowright \overline{M}$ by $x \cdot m = i(x)m - m\,i(x)$

Extend $i : \Lambda^k(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g})^{\otimes k}$

$$x_1 \wedge \dots \wedge x_k \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma \, i(x_{\sigma(1)}) \otimes \dots \otimes i(x_{\sigma(k)})$$

$$\rightsquigarrow \quad \begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}}\left(\mathcal{U}(\mathfrak{g})^{\otimes k}, M\right) & \longrightarrow & \mathrm{Hom}_{\mathbb{C}}\left(\Lambda^k(\mathfrak{g}), \overline{M}\right) \\ \| & & \| \\ C^k\left(\mathcal{U}(\mathfrak{g}); M\right) & \longrightarrow & C^k\left(\mathfrak{g}; \overline{M}\right) \\ \text{(Hochschild)} & & \text{(Chevalley-Eilenberg)} \end{array}$$

is then a quasi-iso. i.e. induces iso. on cohomology groups.