# Lecture 31

0. Chevalley - Eilenberg complex. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and $\mathfrak{g} \circlearrowright M$ be a representation. Recall, $\forall\, k \geq 0$, we defined

$$C^k(\mathfrak{g}; M) = \mathrm{Hom}_{\mathbb{C}}(\Lambda^k \mathfrak{g}, M) = \text{multilinear alternating maps } \mathfrak{g} \times \cdots \times \mathfrak{g} \to M.$$

Differential: $C^{n-1}(\mathfrak{g}; M) \xrightarrow{d} C^n(\mathfrak{g}; M)$ is defined by:

$$d\omega(x_1, \dots, x_n) = \sum_{i=1}^{n} (-1)^{i-1} x_i \cdot \omega(x_1 \dots \hat{x}_i \dots x_n)$$

$x_i$ is skipped.

$$+ \sum_{1 \leq i < j \leq n} (-1)^{i+j} \omega([x_i x_j]\, x_1 \dots \hat{x}_i \dots \hat{x}_j \dots x_n)$$

e.g.

$$\begin{array}{ccccccc} C^0(\mathfrak{g}; M) & \xrightarrow{d} & C^1(\mathfrak{g}; M) & \xrightarrow{d} & C^2(\mathfrak{g}; M) & \to & \cdots \\ \| & & \| & & \| & & \\ M & & \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, M) & & \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g} \wedge \mathfrak{g}, M) & & \end{array}$$

- $m \in M$: $\quad dm : x \longmapsto x \cdot m$
- $f : \mathfrak{g} \to M$: $\quad df : (x_1, x_2) \longmapsto x_1 \cdot f(x_2) - x_2 \cdot f(x_1) - f([x_1, x_2])$
- $\xi : \mathfrak{g} \wedge \mathfrak{g} \to M$: $\quad d\xi(x_1, x_2, x_3) = x_1 \cdot \xi(x_2, x_3) - x_2 \cdot \xi(x_1, x_3) + x_3 \cdot \xi(x_1, x_2) - \xi([x_1 x_2], x_3) + \xi([x_1 x_3], x_2) - \xi([x_2 x_3], x_1)$

1. Some operations on $C^{\bullet}(\mathfrak{g};M)$.

(a) $\forall x \in \mathfrak{g}$, $l_x : C^n \longrightarrow C^{n-1}$ is given by

$$(l_x \omega)(x_1 \dots x_{n-1}) = \omega(x, x_1, \dots, x_{n-1})$$

(b) $\mathcal{L}_x : C^n \longrightarrow C^n$ is given by:

$$(\mathcal{L}_x \omega)(x_1, \dots, x_n) = x \cdot \omega(x_1, \dots, x_n) - \sum_{i=1}^{n} \omega(\dots [x, x_i] \dots)$$

(c) $\lambda(x) : C^n \longrightarrow C^n$ given by

$$(\lambda(x) \cdot \omega)(x_1, \dots, x_n) = x \cdot \omega(x_1, \dots, x_n)$$

<u>Properties</u>: (easy to prove - exercise).

(i) $\{l_x, l_y\} = l_x l_y + l_y l_x = 0 \quad \forall x, y \in \mathfrak{g}$.

(ii) $[\mathcal{L}_x, \mathcal{L}_y] = \mathcal{L}_{[x,y]}$

(iii) $[\lambda(x), \lambda(y)] = \lambda([x,y])$

(iv) $[\mathcal{L}_x, l_y] = l_{[x,y]}$

(v) $[\mathcal{L}_x, d] = 0$

(vi) $d\, l_x + l_x d = \mathcal{L}_x$

2. Now we assume that $\mathfrak{g}$ is simple. Let $C = \sum_a x_a x^a$ be the Casimir element of $\mathfrak{g}$. That is: $\{x_a\}$, $\{x^a\}$ are two bases of $\mathfrak{g}$

such that $(x_a, x^b) = \delta_{ab}$ ; where $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is non-degenerate, symmetric, invariant, bilinear form.

$$(x, [y, z]) = ([x, y], z)$$

Prop. Let $M$ be a f.d. irred. representation of $\mathfrak{g}$. Then

$$H^n(\mathfrak{g}; M) = 0 \qquad \forall\, n \geq 0.$$

Proof. Define $h : C^n(\mathfrak{g}; M) \longrightarrow C^{n-1}(\mathfrak{g}; M)$ by

$$(h\omega)(x_1 \dots x_{n-1}) = \sum_a x_a \cdot \omega(x^a, x_1, \dots, x_{n-1}).$$ That is,

$h = \sum_a \lambda(x_a) L_{x^a}$. <u>Claim</u> : $dh + hd = c_M \cdot \mathrm{Id}$ where $c_M \in \mathbb{C}^\times$ is the scalar by which $C$ acts on $M$ :

$$C \cdot m = c_M \cdot m \qquad \forall\, m \in M.$$ (Here we are using that $M$ is not trivial!)

Given the claim, the proposition follows:

$$d\omega = 0 \Rightarrow dh\omega = c_M \omega \Rightarrow \omega \in \mathrm{Im}(d).$$

Proof of the claim : Unfolding the definitions gives, $\forall\, \omega \in C^n(\mathfrak{g}; M)$

$$(dh + hd)\omega : (y_1, \dots, y_n) \mapsto x_a \cdot x^a \cdot \omega(y_1, \dots, y_n)$$

$$+ \sum_{i=1}^{n} (-1)^i \, [x_a, y_i] \cdot \omega(x^a, y_1, \dots \hat{y_i} \dots y_n) + x_a \cdot \omega([x^a, y_i], y_1, \dots \hat{y_i} \dots y_n)$$

(summation over $a$ is assumed)

Now - if

$$[X_a, Y_i] = \sum_b X_b \cdot \alpha^b_{ai}$$

$$[X^b, Y_i] = \sum_a X^a \cdot \beta^{bi}_a$$

$$\boxed{\begin{aligned}\alpha^b_{ai} &= ([X_a, Y_i], X^b) \\ &= (X_a, [Y_i, X^b]) = -\beta^{bi}_a\end{aligned}}$$

then the $i^{\text{th}}$ term in the last sum is :

$$(-1)^i \underbrace{\left(\alpha^b_{ai} + \beta^{bi}_a\right)}_{0} \cdot X_b \cdot \omega(X^a, Y_1, \dots \hat{Y}_i \dots Y_n) = 0. \qquad \square$$

3. Now assume $N$ is a trivial $\mathfrak{g}$ - representation.

**Prop.** $H^n(\mathfrak{g}; N) \cong \mathrm{Hom}_{\mathfrak{g}}(\Lambda^n \mathfrak{g}, N)$

**Proof.-** Recall $\mathfrak{g}$ acts on $\mathrm{Hom}_{\mathbb{C}}(\Lambda^n \mathfrak{g}, N)$ via $\mathcal{L}_x$ (item (b) of §1 above). and commutes with the differential. (item (v) of Properties from §1.). Note, also, that since $\mathfrak{g} \circlearrowright N$ is trivial,

$$d\Big|_{\mathrm{Hom}_{\mathfrak{g}}(\Lambda^n \mathfrak{g}, N)} \equiv 0 : \quad \forall\, \omega \in \mathrm{Hom}_{\mathfrak{g}}(\Lambda^n \mathfrak{g}, N) \subset \underbrace{\mathrm{Hom}_{\mathbb{C}}(\Lambda^n \mathfrak{g}; N)}_{C^n(\mathfrak{g}; N)}$$

$$d\omega(Y_1, \dots, Y_{n+1}) = \sum_{i<j} (-1)^{i+j} \omega([Y_i, Y_j], Y_1 \dots \hat{Y}_i \dots \hat{Y}_j \dots Y_{n+1})$$

$$= \frac{1}{2} \sum_i (Y_i \cdot \omega)(Y_1 \dots \hat{Y}_i \dots Y_{n+1}) = 0.$$

Define $h = \sum_a \mathcal{L}_{X_a} \iota_{X^a} : C^{n+1}(\mathfrak{g};N) \to C^n(\mathfrak{g};N)$

Claim : $dh + hd = \mathcal{L}_{X_a}\mathcal{L}_{X^a}$

= action of the Casimir element on $C^n(\mathfrak{g};N)$

[Pf. of the claim : $dh + hd = d\mathcal{L}_{X_a}\iota_{X^a} + \mathcal{L}_{X_a}\iota_{X^a} d$

$= \mathcal{L}_{X_a}(d\iota_{X^a} + \iota_{X^a} d)$ using (v) - page 2

$= \mathcal{L}_{X_a}\mathcal{L}_{X^a}$ using (vi) □ ]

Given the claim, the proposition follows since $\mathrm{Hom}_{\mathfrak{g}}(\Lambda^n\mathfrak{g}, N)$

$= C^n(\mathfrak{g};N)^{\mathfrak{g}}$ - where Casimir acts as zero.

$C^n(\mathfrak{g};N) = \mathrm{Hom}_{\mathfrak{g}}(\Lambda^n\mathfrak{g}, N) \oplus \boxed{\overline{C}^n(\mathfrak{g};N)}$

Casimir acts by non-zero scalars - and $h$ provides the homotopy - as in the previous proposition. □

4. Corollary - $\mathfrak{g} \circlearrowright M$ finite-dimensional ; or more generally, a direct sum of f.d. repns ; Then (possibly $\infty$) $\boxed{\text{Whitehead's Lemma}}$

$H^n(\mathfrak{g};M) = 0$ for $n = 1, 2$.

Proof. – Combining Props 2 and 3 above :

$$H^n(\mathfrak{g};M) \cong (\Lambda^n \mathfrak{g}^*)^{\mathfrak{g}} \otimes M^{\mathfrak{g}}.$$ Now we have

$n=1$ : $(\mathfrak{g}^*)^{\mathfrak{g}} = (0)$ since $\mathfrak{g}$ is simple.

$n=2$ : $(\mathfrak{g}^* \wedge \mathfrak{g}^*)^{\mathfrak{g}} = (0)$ ( $(\mathfrak{g}\otimes\mathfrak{g})^{\mathfrak{g}}$ is 1-dim'l spanned by the Casimir tensor which is symmetric not skew-symm.) □

5. Taking $M = \mathcal{U}(\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} S^n(\mathfrak{g})$ we get (as $\mathfrak{g}$-repns)

$HH^n(\mathcal{U}(\mathfrak{g});\mathcal{U}(\mathfrak{g})) = 0$ for $n=1$ and $2$. Hence

(5.1) $\exists$ an alg. iso. $\psi : \mathcal{U}_\hbar(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})[[\hbar]]$ s.t. $\psi = \mathrm{id} \pmod{\hbar}$

( i.e. $\overline{\psi} : \mathcal{U}_\hbar(\mathfrak{g})/\hbar\,\mathcal{U}_\hbar(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{g})$
$H_i \longmapsto h_i$ ; $E_i \mapsto e_i$ ; $F_i \mapsto f_i$ )

(5.2) If $\psi_1, \psi_2 : \mathcal{U}_\hbar(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})[[\hbar]]$ are two alg. iso-s then $\exists$ $a \in 1 + \hbar\,\mathcal{U}(\mathfrak{g})[[\hbar]]$ s.t.

$$\psi_1(x) = a\,\psi_2(x)\,a^{-1}.$$