Lecture 32

0. Recall that we are aiming to prove the Kohno-Drinfeld theorem:

$$\left(U_\hbar\mathfrak{g},\ \Delta,\ R,\ 1^{\otimes 3}\right) \underset{\substack{\text{twist}\\ \text{equiv.}}}{\sim} \left(U\mathfrak{g}[[\hbar]],\ \Delta_0,\ R_{KZ} = e^{\frac{\hbar}{2}\Omega},\ \Phi_{KZ}\right).$$

Last time we showed that:

(i) $\exists$ an algebra iso. $\psi : U_\hbar\mathfrak{g} \longrightarrow U(\mathfrak{g})[[\hbar]]$ s.t. $\psi = \mathrm{id} \pmod{\hbar}$.

(ii) any two algebra isomorphisms $\psi_1, \psi_2 : U_\hbar(\mathfrak{g}) \to U(\mathfrak{g})[[\hbar]]$ are related by: $\exists\ a = 1 + \hbar(\cdots) \in U(\mathfrak{g})[[\hbar]]$ s.t.

$$\psi_2(x) = a\,\psi_1(x)\,a^{-1}.$$

Corollary. The centers of $U_\hbar(\mathfrak{g})$ and $U(\mathfrak{g})[[\hbar]]$ are canonically isomorphic: $Z(U_\hbar(\mathfrak{g})) \cong Z(U(\mathfrak{g}))[[\hbar]]$.

$\left(Z(A) = \{a \in A : ab = ba\ \forall\, b \in A\} \text{ for an assoc. alg. } A \right)$

Definition: $C_\hbar \in U_\hbar(\mathfrak{g})$ is defined to be the unique element of $Z(U_\hbar(\mathfrak{g}))$ s.t. $\psi(C_\hbar) = C \left(= \underbrace{\sum_a x_a x^a \in U(\mathfrak{g})}_{\text{Casimir element}} \right)$

Note: by the corollary above (or (ii) & the fact that $C \in Z(U(\mathfrak{g}))$) this definition of quantum Casimir element is independent of the identification $\psi : U_\hbar(\mathfrak{g}) \cong U(\mathfrak{g})[[\hbar]]$.

The following proposition will be used today, and its proof will be given later: let $\theta = q^{C_\hbar} = e^{\frac{\hbar}{2}C_\hbar} \in U_\hbar(\mathfrak{g})$.

1. Proposition. – Let $\mathcal{R}$ be the universal R-matrix of $U_\hbar(\mathfrak{g})$.

Then $$\Delta(\theta)\cdot\theta^{-1}\otimes\theta^{-1} = \mathcal{R}_{21}\cdot\mathcal{R}$$
$$= \Delta\left(e^{\hbar C_\hbar/2}\right)\cdot e^{-\frac{\hbar}{2}C_\hbar}\otimes e^{-\frac{\hbar}{2}C_\hbar}$$

Remark. – This result is due to Drinfeld (On almost cocomm...) and is also true for $R_{KZ} = e^{\hbar/2\cdot\Omega}$ – since

$$\underset{0}{\Delta}(C) = \sum_a (x_a\otimes 1 + 1\otimes x_a)(x^a\otimes 1 + 1\otimes x^a)$$
$$= C\otimes 1 + 1\otimes C + 2\cdot\Omega$$
$$\Rightarrow \quad \underset{0}{\Delta}\left(e^{\frac{\hbar}{2}C}\right) = \left(e^{\frac{\hbar}{2}C}\otimes e^{\frac{\hbar}{2}C}\right)\cdot e^{\hbar\Omega}$$
$$\Rightarrow \quad (R_{KZ})_{21}\cdot R_{KZ} = e^{\hbar\Omega} = \underset{0}{\Delta}\left(e^{\frac{\hbar}{2}C}\right)\cdot e^{-\frac{\hbar}{2}C}\otimes e^{-\frac{\hbar}{2}C}.$$

2. Matching the coproduct.

Using $\psi: U_\hbar\mathfrak{g}\longrightarrow U(\mathfrak{g})[[\hbar]]$, we obtain another coproduct on $U(\mathfrak{g})[[\hbar]]$, denoted by Δ^ψ, defined as:

$$\Delta^\psi(x) = \psi\otimes\psi\left(\Delta(\psi^{-1}(x))\right).$$

It is clear from the definition of Δ and the fact that $\psi = id \pmod{\hbar}$ that $$\Delta^\psi(x) = \underset{0}{\Delta}(x) + \hbar(\ldots)$$
$$\underset{(\text{Notation})}{=} \underset{0}{\Delta}(x) + \sum_{n\geq 1} \mathcal{D}_n(x)\hbar^n \quad \text{where, } \mathcal{D}_n: U\mathfrak{g}\to(U\mathfrak{g})^{\otimes 2}.$$

Prop. – There exists $F \in \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}[[\hbar]]$, $F = 1 \otimes 1 \pmod{\hbar}$

s.t. $\Delta^{\psi}(x) = F \cdot \Delta_0(x) \cdot F^{-1}$.

Proof. The proof uses the same type of arguments we saw in Lecture 30 (see prop. 3 on page 4 of L. 30) and the fact that $HH^1(\mathcal{U}(\mathfrak{g}); \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})) = 0$.

We view $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ as a bimodule over $\mathcal{U}(\mathfrak{g})$ via Δ_0:

$$x \cdot (A) = \Delta_0(x) A \quad ; \quad (A) \cdot x = A \Delta_0(x) \quad \forall x \in \mathcal{U}(\mathfrak{g}),\ A \in \mathcal{U}(\mathfrak{g})^{\otimes 2}.$$

Since $\Delta^{\psi} = \sum_{n \geq 0} \hbar^n \mathcal{D}_n$ is an algebra hom & $\mathcal{D}_0 = \Delta_0$, we get

$$\Delta^{\psi}(ab) = \Delta^{\psi}(a) \cdot \Delta^{\psi}(b) \underset{\text{Coeff of } \hbar}{\rightsquigarrow} \mathcal{D}_1(ab) = \Delta_0(a) \cdot \mathcal{D}_1(b) + \mathcal{D}_1(a) \Delta_0(b)$$

$(\forall a, b \in \mathcal{U}(\mathfrak{g}))$

i.e. $\mathcal{D}_1 \in \underset{\substack{\uparrow \\ \mathrm{Ker}(d_2)}}{Z^1}(\mathcal{U}(\mathfrak{g}); \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})) = \underset{\substack{\uparrow \\ \mathrm{Im}(d_1)}}{B^1}(\mathcal{U}(\mathfrak{g}); \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))$

So $\mathcal{D}_1 = d_1(A)$ for some $A \in C^0(\mathcal{U}(\mathfrak{g}); \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})) = \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$

(i.e. $\mathcal{D}_1(x) = \Delta_0(x) A - A \Delta_0(x)$)

$\Rightarrow$ Conjugating by $1 + \hbar A$, we arrive at $\Delta_0 + O(\hbar^2)$.

Proposition is proved by repeating this argument. □

3. In conclusion:

$$\left(\mathcal{U}_\hbar(\mathfrak{g}),\ \Delta,\ \mathcal{R},\ 1^{\otimes 3}\right) \underset{\substack{\text{using}\\ \psi}}{\rightsquigarrow} \left(\mathcal{U}(\mathfrak{g})[\![\hbar]\!],\ \Delta^{\psi},\ \psi\otimes\psi(\mathcal{R}),\ 1^{\otimes 3}\right)$$

$$\underset{\substack{\text{twist by}\\ F \text{ from Prop. 2}\\ \text{above}}}{\rightsquigarrow} \left(\mathcal{U}(\mathfrak{g})[\![\hbar]\!],\ \Delta_0,\ \underset{\substack{\uparrow\\ 1\otimes 1+\hbar(\cdots)}}{\mathcal{R}'},\ \Phi'\right)$$

4. Matching the R-matrix.

Proposition: There exists $\tilde{F} \in \mathcal{U}(\mathfrak{g})^{\otimes 2}[\![\hbar]\!]$ $(= 1^{\otimes 2} \bmod \hbar)$ s.t.

- $\tilde{F}_{21}\mathcal{R}'\tilde{F}$ is symmetric
- $\tilde{F}$ is invariant $(\tilde{F}\Delta_0(x) = \Delta_0(x)\tilde{F}\ \ \forall x \in \mathcal{U}\mathfrak{g})$

Proof. This step holds in more generality: let (H, Δ, R, Φ) be a quasi-triangular quasi bialgebra (see Lecture 20) over $\mathbb{C}[\![\hbar]\!]$ s.t. $R = 1\otimes 1 \pmod{\hbar}$. We will prove the existence of F s.t. $R_F = F_{21}\cdot R\cdot F^{-1}$ is symmetric.

i.e. $\quad R_F = (R_F)_{21}$ i.e. $F_{21}\cdot R\cdot F^{-1} = F\cdot R_{21}\cdot F_{21}^{-1}$

$$\Longleftrightarrow \quad R_{21} = F^{-1}F_{21}\cdot R\cdot F^{-1}F_{21}$$

Thus $\quad R_{21}\cdot R = (F^{-1}F_{21}\cdot R)^2$

$$F^{-1}F_{21}R = (R_{21}\cdot R)^{1/2} \quad \text{or} \quad F^{-1}F_{21} = (R_{21}R)^{\frac{1}{2}}R^{-1}$$

Thus, we want to find F s.t.

$$F_{21}^{-1} F = R(R_{21}R)^{-1/2} =: \rho$$

$\boxed{\text{Claim : } \rho_{21}\rho = 1}$. Given the claim, $F := \rho^{\frac{1}{2}}$ solves the equation above. If, in addition, $\Delta = \Delta^{op}$ for (H, Δ, R, Φ), then both R and R_{21} have to be invariant and hence, so will be ρ and F.

<u>Proof of the claim</u>. - Note that $(RR_{21})^n \cdot R = R \cdot (R_{21}R)^n \ \forall n$

$$\Rightarrow \quad (\text{formal series in } RR_{21}) \cdot R$$

$$= R \cdot (\text{same series in } R_{21} \cdot R)$$

$$\Rightarrow \quad (RR_{21})^{-\frac{1}{2}} \cdot R = R \cdot (R_{21}R)^{-\frac{1}{2}}.$$ And hence:

$$\begin{aligned}\rho_{21}\rho &= R_{21}(RR_{21})^{-\frac{1}{2}} \cdot R(R_{21}R)^{-\frac{1}{2}} \\ &= R_{21} \cdot R \cdot (R_{21}R)^{-\frac{1}{2}}(R_{21}R)^{-\frac{1}{2}} = (R_{21}R)(R_{21}R)^{-1} \\ &= 1.\end{aligned}$$

$\square$

5. Returning back to the diagram from §3 above:

$$(\mathcal{U}_\hbar(\mathfrak{g}), \Delta, R, 1) \rightsquigarrow (\mathcal{U}(\mathfrak{g})[\![\hbar]\!], \Delta^\psi, \psi \otimes \psi(R), 1)$$

$$\underset{\substack{\text{twist by} \\ J = \tilde{F} \cdot F \ (\text{props: } 2 \,\&\, 4 \text{ above})}}{\rightsquigarrow} (\mathcal{U}(\mathfrak{g})[\![\hbar]\!], \Delta_0, \tilde{R}, \tilde{\Phi}).$$

this is symmetric $\tilde{R}_{21} = \tilde{R}$.

Proposition $\quad \widetilde{R} = R_{KZ} = e^{\frac{\hbar}{2}\cdot\Omega}.$

Proof

$$(\widetilde{R})^2 = \widetilde{R}_{21}\cdot\widetilde{R}$$

$$= J\cdot\psi\otimes\psi(\mathcal{R}_{21}\cdot\mathcal{R})\cdot J^{-1}$$

(see §1 on page 2 above)

$$= J\cdot\psi\otimes\psi\left(\Delta(q^{C_\hbar})\cdot q^{-C_\hbar}\otimes q^{-C_\hbar}\right) J^{-1}$$

$$= \boxed{J\cdot\Delta^{\psi}(e^{\frac{\hbar}{2}C})\cdot J^{-1}}\; e^{-\frac{\hbar}{2}C}\otimes e^{-\frac{\hbar}{2}C}$$

$$= \Delta_0(e^{\frac{\hbar}{2}C})\cdot e^{-\frac{\hbar}{2}C}\otimes e^{-\frac{\hbar}{2}C} = e^{\hbar\Omega}$$

$$\Rightarrow \quad \widetilde{R} = e^{\frac{\hbar}{2}\Omega} = R_{KZ}$$

$\square$

$q^C\otimes q^C$ commutes with any $J\in\mathcal{U}(\mathfrak{g})^{\otimes 2}[[\hbar]]$