Lecture 33

0. Today's theorem. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Assume that there are two quasi-bialgebras:

$$\left(U\mathfrak{g}[\![\hbar]\!], \underset{0}{\Delta}, R, \Phi\right) \qquad \left(U\mathfrak{g}[\![\hbar]\!], \underset{0}{\Delta}, R, \Phi'\right)$$

$$\boxed{\begin{array}{l} \text{where } \quad R = R_{KZ} = e^{\frac{\hbar}{2}\Omega} \\ \text{and } \Phi, \Phi' \equiv 1^{\otimes 3} \mod \hbar \end{array}}$$

. Then there exists a twist

$$F \in 1 \otimes 1 + \hbar\left(U\mathfrak{g} \otimes U\mathfrak{g}[\![\hbar]\!]\right)$$

s.t. (i) $F = F_{21}$ (ii) $[F, \underset{0}{\Delta}(a)] = 0 \quad \forall\, a \in U\mathfrak{g}$

(iii) $\Phi' = \Phi_F$ (associator Φ twisted by F)

$$= 1 \otimes F \cdot \mathrm{id} \otimes \Delta(F) \cdot \Phi \cdot \Delta \otimes \mathrm{id}\,(F^{-1}) \cdot F^{-1} \otimes 1$$

Note - $\underset{0}{\Delta}$ and R remain unchanged upon twisting with such an F:

$$\underset{F}{\Delta}(x) = F\, \underset{0}{\Delta}(x)\, F^{-1}$$
$$= \underset{0}{\Delta}(x) \qquad \text{by (ii)}$$

$$R_F = F_{21} \cdot R \cdot F^{-1} = F \cdot e^{\frac{\hbar}{2}\Omega} F^{-1} \quad \text{by (i)}$$
$$= e^{\frac{\hbar}{2}\Omega} \qquad \text{since } \Omega \text{ is the Casimir tensor - commutes with } F.$$

This theorem appeared in V. Drinfeld - On quasi-triangular quasi Hopf algebras and a group closely related to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -(1990).

The proof of the theorem uses, in an essential way, the cohomology of <u>cobar complex</u> which was defined by Adams (1956) and Cartier (1957) independently.

1. Cobar complex. Let C be a coalgebra – that is, C is a vector space over $\mathbb{C}$; $\Delta : C \to C \otimes C$ linear map which is coassociative $(\Delta \otimes id \circ \Delta = id \otimes \Delta \circ \Delta)$; $\varepsilon : C \to \mathbb{C}$ linear form s.t. $\varepsilon \otimes id(\Delta(x)) = x = id \otimes \varepsilon(\Delta(x)) \ \forall x \in C$.

We assume that $\exists$ a distinguished element $1_C \in C$ s.t.

$\Delta(1_C) = 1_C \otimes 1_C$, and hence $\varepsilon(1_C) = 1 \in \mathbb{C}$.

The data of $(C, \Delta, \varepsilon, 1_C)$ is called <u>augmented</u> <u>coalgebra</u>.

Define a complex (TC, δ) as follows:

$$TC_n = \underbrace{C \otimes \dots \otimes C}_{n\text{-times}} \quad (\text{or } T^nC) \ ; \ T^0C = \mathbb{C}$$

$$\delta : T^nC \longrightarrow T^{n+1}C$$

$$x_1 \otimes \dots \otimes x_n \longmapsto 1_C \otimes x_1 \otimes \dots \otimes x_n + \sum_{i=1}^{n} (-1)^i x_1 \otimes \dots \Delta(x_i) \dots \otimes x_n$$
$$+ (-1)^{n+1} x_1 \otimes \dots \otimes x_n \otimes 1_C$$

$(n=0)$ $\delta : \mathbb{C} \to T^1C = C$ is identically 0.

Check : $\delta \circ \delta = 0$.

(TC, δ) is called the cobar complex of the augmented coalgebra C.

We will be interested in the special case when $C = U\mathfrak{g}$

2. Motivation. - If C is a quasi-bialgebra with an associator

$\Phi \in C^{\otimes 3}[\![\hbar]\!]$; $\Phi = 1^{\otimes 3} + \hbar \varphi + \cdots$

then the pentagon axiom :

$$(1 \otimes \Phi) \cdot (id \otimes \Delta \otimes id)(\Phi) \cdot (\Phi \otimes 1)$$
$$= (id \otimes id \otimes \Delta)(\Phi) \cdot (\Delta \otimes id \otimes id)(\Phi)$$

$\Rightarrow$

Coeff of $\hbar$

$$1 \otimes \varphi + id \otimes \Delta \otimes id(\varphi) + \varphi \otimes 1$$
$$- \Delta \otimes id \otimes id(\varphi) - id \otimes id \otimes \Delta(\varphi) = 0$$

i.e. $\delta \varphi = 0$. So φ is a 3-cocycle in the cobar complex.

Similarly if $\Phi' = 1 \otimes F \cdot id \otimes \Delta(F) \cdot \Phi \cdot \Delta \otimes id(F^{-1}) \cdot F^{-1} \otimes 1$

$F = 1 \otimes 1 + \hbar f + \cdots \in C \otimes C[\![\hbar]\!]$

$\rightsquigarrow$

coeff of $\hbar$

$$\varphi' - \varphi = 1 \otimes f - \Delta \otimes id(f) + id \otimes \Delta(f) - f \otimes 1$$
$$= \delta f$$

i.e. φ and φ' differ by a coboundary.

3. Further structures on (TC, δ) :

(a) $T^nC \times T^mC \xrightarrow{\bullet} T^{n+m}C$ concatenation of tensors

satisfies $\boxed{\delta(a \cdot b) = \delta(a) \cdot b + (-1)^n a \cdot \delta(b)}$

i.e. (TC, δ) is a differential graded algebra.

(b) Assume Δ is cocommutative $(\Delta^{op} = \Delta)$. Then we have an involution, σ

$$\sigma(x_1 \otimes \ldots \otimes x_n) = (-1)^{\frac{n(n+1)}{2}} x_n \otimes \ldots \otimes x_1 .$$

This involution commutes with δ (easy exercise) and hence

$$(TC, \delta) = (T_+C, \delta) \oplus (T_-C, \delta)$$

$$\{\xi \in TC : \sigma\xi = \xi\} \qquad \{\xi \in TC : \sigma\xi = -\xi\}$$

(c) (Still assuming $\Delta^{op} = \Delta$) let there be a simple Lie algebra $\mathfrak{g}$ acting on C s.t.

$$(1) \begin{cases} \Delta(x \cdot c) = (x \otimes 1 + 1 \otimes x) \cdot \Delta(c) & \forall x \in \mathfrak{g},\ c \in C. \\ x \cdot 1_C = 0 \end{cases}$$

(2) C = a direct sum of f.d. repns. of $\mathfrak{g}$.

Then $(T_\pm C, \delta) = ((T_\pm C)^{\mathfrak{g}}, \delta) \oplus (\mathfrak{g} \cdot T_\pm C, \delta)$

4. By a more precise version of the PBW theorem:

$$\mathrm{Sym}(\mathfrak{g}) \xrightarrow{\text{symmetrization}} T(\mathfrak{g}) \xrightarrow{\text{canonical surjection}} \mathcal{U}(\mathfrak{g}) = T(\mathfrak{g}) \big/ \text{a 2-sided ideal}$$

$$x_1 \dots x_n \longmapsto \frac{1}{n!} \sum_{\pi \in S_n} x_{\pi(1)} \otimes \dots \otimes x_{\pi(n)}$$

$\leadsto$ $\mathrm{Sym}(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{g})$ as $\mathfrak{g}$-repns and coalgebras (augmented)

This makes the following computation of cohomology particularly relevant.

Theorem (Cartier 1957). Let V be a finite-dimensional vector space (over $\mathbb{C}$). Taking C = augmented coalgebra $\mathrm{Sym}(V)$, we have

$$H^{2n}(T_+C) = \Lambda^{2n} V \; ; \; H^{2n+1}(T_+C) = 0$$

$$H^{2n}(T_-C) = 0 \; ; \; H^{2n+1}(T_-C) = \Lambda^{2n+1} V.$$

This theorem is proved by constructing a homomorphism of differential graded algebras

$$\mu : (T\,\mathrm{Sym}V, \delta) \to (\Lambda^{\bullet} V, 0)$$

which induces iso on cohomology (i.e. is a quasi-iso.):

Let $\mathrm{pr}_V : \mathrm{Sym}V \to V$ be given by projection onto the linear term:

$$\begin{array}{ccc} \mathrm{Sym}\,V & = & \mathbb{C} \oplus V \oplus \mathrm{Sym}^2(V) \oplus \cdots \\ \mathrm{pr}_V \downarrow & & \\ V & & \end{array}$$

project onto this component

Then $\mu : T^n \,\mathrm{Sym}\,V \longrightarrow \Lambda^n V$

$$\xi_1 \otimes \cdots \otimes \xi_n \longmapsto \mathrm{pr}_V(\xi_1) \wedge \cdots \wedge \mathrm{pr}_V(\xi_n)$$

5. Proof of Theorem §0.

Idea. Consider the difference $\Phi - \Phi' = \hbar^n \varphi + \hbar^{n+1}(\cdots)$ $(n \geq 1)$.

Then we claim that φ satisfies:

true since Δ is coassoc.

$\Rightarrow \Phi, \Phi'$ are invariant

easy. see §2 above

- φ is $\mathfrak{g}$-invariant
- $\varphi_{321} = -\varphi$
- $\delta\varphi = 0.$
- $\mathrm{Alt}(\varphi) = \sum_{\pi \in S_3} (-1)^\pi \, \pi . \varphi = 0$

That is $\varphi \in (T^3_- C)^{\mathfrak{g}}$ and $\delta\varphi = 0.$ $\quad (C = U\mathfrak{g} \cong \mathrm{Sym}(\mathfrak{g}))$

By Cartier's Theorem, $\varphi = \delta f$ for some $f \in (T^2_- C)^{\mathfrak{g}}$

$\mathfrak{g}$-invariant and

$$f = f_{21} \Leftarrow \begin{cases} \sigma(f) = -f & (\text{defn. of } T_- C) \\ \quad \| & \\ -f_{21} & \\ (\text{defn of } \sigma) & \end{cases}$$

Thus twisting Φ by $F = 1\otimes 1 + \hbar^n f$ gives $\Phi_F \equiv \Phi' \mod \hbar^{n+1}$.

6. Proofs of $\varphi_{321} = -\varphi$ and $Alt(\varphi) = 0$.

(6.1) Lemma. $\Phi_{321} = \Phi^{-1}$ (also for Φ') $\left(\text{only uses } \underset{\circ}{\Delta} = \underset{\circ}{\Delta}^{op} \text{ and } R_{21} = R\right)$.

Proof. We claim that (left as an exercise)* (see next page)

$$R_{12} \cdot \Delta\otimes id\,(R) = \Phi_{321} \cdot R_{23} \cdot id\otimes\Delta\,(R)\cdot\Phi \quad -(1)$$

$$\Longrightarrow \quad R_{23} \cdot id\otimes\Delta\,(R) = \Phi \cdot R_{12} \cdot \Delta\otimes id\,(R)\cdot\Phi_{321} \quad -(2)$$

(apply (13) and use $\Delta = \Delta^{op}$, $R = R_{21}$)

Combining (1) and (2):

$$\left(R_{12}\cdot\Delta\otimes id\,(R)\right)^2 = \Phi^{-1}\left(R_{23}\; id\otimes\Delta\,(R)\right)^2\Phi$$

$$\Longrightarrow \quad R_{12}\cdot\Delta\otimes id\,(R) = \Phi^{-1}\cdot\left(R_{23}\cdot(id\otimes\Delta)(R)\right)\Phi$$

by uniqueness of square root

Compare this with (1) to get $\Phi_{321} = \Phi^{-1}$. $\square$

Cor. $\varphi_{321} = -\varphi$

Pf. Modulo $\hbar^{n+1}$:

$$\Phi'_{321} \equiv \Phi_{321} + \hbar^n\varphi_{321}$$

$$(\Phi')^{-1} \equiv \Phi^{-1} - \hbar^n\varphi \qquad \square$$

(6.2) Lemma. $\mathrm{Alt}(\varphi) = \varphi - \varphi_{213} - \varphi_{132} - \varphi_{321} + \varphi_{231} + \varphi_{312} = 0$

Proof. Consider the hexagon axiom

$$\Delta \otimes id\,(R) = \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi$$

$$= \Phi'_{312} R_{13} \left(\Phi'_{132}\right)^{-1} R_{23} \Phi'$$

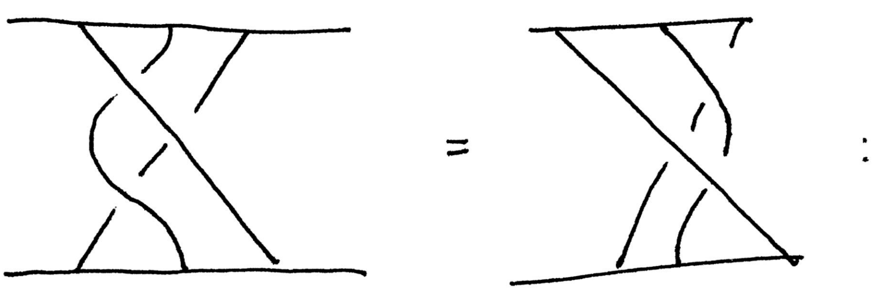
$\rightsquigarrow$ mod. $\hbar^{n+1}$

$$\varphi_{312} - \varphi_{132} + \varphi = 0$$

$$-\varphi_{213} + \varphi_{231} - \varphi_{321} = 0$$

(either using the other hexagon axiom; or $\varphi_{321} = -\varphi$ and the previous line)

Sum: $\mathrm{Alt}(\varphi) = 0$. $\square$



(Yang-Baxter eq^n)

$$\underbrace{R_{12} \Phi_{312} R_{13} \Phi^{-1}_{132} R_{23} \Phi_{123}}_{\overset{\text{HEX.1}}{=}\ R_{12}\ \Delta \otimes id\,(R)} = \underbrace{\Phi_{321} R_{23} \Phi^{-1}_{231} R_{13} \Phi_{213} R_{12}}_{\overset{\text{HEX.2}}{=}\ \Phi_{321} R_{23}\ id \otimes \Delta(R)\ \Phi}$$