Lecture 34

0. Summary so far. – Recall that we finished a proof of Kohno-Drinfeld theorem, modulo two results (recalled below), along the following lines:

$$(U_\hbar(\mathfrak{g}), \Delta, R, 1^{\otimes 3}) \overset{①}{\rightsquigarrow} (U\mathfrak{g}[[\hbar]], \Delta^\psi, \psi\otimes\psi(R), 1^{\otimes 3})$$

$\boxed{\exists \text{ alg iso } \psi : U_\hbar(\mathfrak{g}) \to U\mathfrak{g}[[\hbar]] \quad \psi = id \bmod \hbar}$

$$\overset{②}{\rightsquigarrow} (U\mathfrak{g}[[\hbar]], \overset{\circ}{\Delta}, R_{KZ}, \Phi) \qquad \left(\begin{array}{l}\exists\, F = 1^{\otimes 2} + \hbar(\dots) \in U\mathfrak{g}^{\otimes 2}[[\hbar]] \\ \text{s.t. } \Delta^\psi(x) = F\,\overset{\circ}{\Delta}(x)\,F^{-1} \\ \psi\otimes\psi(R) = F_{21}\, R_{KZ}\, F^{-1}\end{array}\right)$$

Finally, $\Phi \overset{③}{\sim} \Phi_{KZ}$.

- ① and part of ② used Whitehead's Lemmas:

  $HH^i(U\mathfrak{g}\,;\, U\mathfrak{g}) = 0$; $HH^1(U\mathfrak{g}, U\mathfrak{g}^{\otimes 2}) = 0$

  $i = 1, 2$

- ② also used an element $\theta = q^{C_\hbar}$ s.t.

  $R_{21} \cdot R = \Delta(\theta) \cdot \theta^{-1} \otimes \theta^{-1}$ in $U_\hbar(\mathfrak{g})^{\otimes 2}$. (To be proved - next lecture)

- ③ used cohomology of coalgebras - <u>Cartier's theorem</u>

  (To be proved - today)

1. Let V be a finite-dimensional vector space (over $\mathbb{C}$). Consider the following coalgebra $C = Sym(V)$ or just $S(V)$:

- $\Delta : C \longrightarrow C \otimes C$ is defined by :

$$\Delta(1) = 1 \otimes 1 \;; \quad \Delta(v) = v \otimes 1 + 1 \otimes v \quad \forall\, v \in V = S^1(V)$$

$(1 \in \mathbb{C} = S^0 V \subset C)$

- $\varepsilon : C \to \mathbb{C}$ is projection along $S^0 V = \mathbb{C}$.

The cobar complex $(T^\bullet C, \delta)$ was defined in the last lecture :

$$0 \to \mathbb{C} = T^0 C \xrightarrow{\delta} C \xrightarrow{\delta} C \otimes C \xrightarrow{\delta} \cdots \to \underset{C^{\otimes n}}{\overset{\shortparallel}{T^n C}} \xrightarrow{\delta} \underset{C^{\otimes(n+1)}}{\overset{\shortparallel}{T^{n+1} C}} \to \cdots$$

$\delta : T^n C \to T^{n+1} C$ is zero for $n = 0$; otherwise

$$\delta(p_1 \otimes \cdots \otimes p_n) = 1 \otimes p_1 \otimes \cdots \otimes p_n + \sum_{i=1}^{n} (-1)^i \cdots \Delta(p_i) \cdots$$
$$+ (-1)^{n+1} p_1 \otimes \cdots \otimes p_n \otimes 1 .$$

Example $n = 1$. For $p \in C = \mathrm{Sym}\, V$, $\delta(p) = 1 \otimes p - \Delta(p) + p \otimes 1$.

One can easily prove that $\delta(p) = 0 \iff p \in V$

In general $\delta(p) = 0$ then means $\Delta(p) = p \otimes 1 + 1 \otimes p$, i.e.

$p \in C$ is <u>primitive</u>.

$$H^1(T^\bullet C, \delta) = \{ p \in C \mid \Delta(p) = p \otimes 1 + 1 \otimes p \} \cong V = \Lambda^1 V.$$

Recall : $T^n \mathbb{K} \times T^m \mathbb{K} \to T^{n+m} \mathbb{K}$ makes $(T^\bullet C, \delta)$ into

a differential graded algebra

2. Consider $pr_V : C = \mathrm{Sym}\, V \longrightarrow V$ (projection onto $S^1 V \cong V$)

which we extended to $\mu : T^n C \to \Lambda^n V$.

Theorem. (a) $\mu : (T^\bullet C, \delta) \longrightarrow (\Lambda^\bullet V, 0)$ is a chain map, in fact a morphism of differential graded algebras.

The induced map on cohomology is an isomorphism.

(b) Let $\alpha : \Lambda^n V \to T^n C$ be the antisymmetrization map:

$$\alpha(v_1 \wedge \ldots \wedge v_n) = \sum_{\sigma \in S_n} (-1)^\sigma v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} .$$

Then $\delta \circ \alpha = 0$ and $\mu(\alpha(\omega)) = n!\, \omega \quad \forall\, \omega \in \Lambda^n V$.

Hence $H^n(T^\bullet C, \delta) \cong \Lambda^n V$ via

$$\Lambda^n V \xrightarrow{\alpha} \mathrm{Ker}(\delta : T^n \to T^{n+1}) \twoheadrightarrow H^n(T^\bullet C, \delta).$$

Under the involution $p_1 \otimes \cdots \otimes p_n \overset{\sigma}{\longmapsto} (-1)^{\frac{n(n+1)}{2}} p_n \otimes \cdots \otimes p_1$, we get

$$\mu \circ \sigma \circ \alpha = (-1)^n \cdot n! .$$

This implies

$$H^{2n}(T_+ C, \delta) \cong \Lambda^{2n} V,$$
$$H^{2n+1}(T_+ C, \delta) = 0,$$
$$H^{2n}(T_- C, \delta) = 0,$$
$$H^{2n+1}(T_- C, \delta) = \Lambda^{2n+1} V.$$

3. Dual statement to that of Theorem 2.

$W = V^*$. The transpose of δ gives us

$A = S^\cdot(W)$

- $$d : T^n A \longrightarrow T^{n-1} A$$

$$\begin{aligned} a_1 \otimes \dots \otimes a_n \longmapsto\ & \varepsilon(a_1) a_2 \otimes \dots \otimes a_n \\ & + \sum_{i=1}^{n-1} (-1)^i a_1 \otimes \dots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \dots \otimes a_n \\ & + (-1)^n a_1 \otimes \dots \otimes a_{n-1} \varepsilon(a_n) \end{aligned}$$

- $$\alpha : \Lambda^n W \longrightarrow T^n A$$

$$w_1 \wedge \dots \wedge w_n \longmapsto \sum_{\sigma \in S_n} (-1)^\sigma w_{\sigma(1)} \otimes \dots \otimes w_{\sigma(n)}$$

- $\mu : T^n A \longrightarrow \Lambda^n W$ as before

Theorem. We have $d \circ \alpha = 0$. The induced map gives an iso. $\forall n \geq 0$

$$\alpha : \Lambda^n W \to H_n(T^\cdot A, d).$$

μ is a chain map and $\mu \circ \alpha(\omega) = n! \cdot \omega \quad \forall \omega \in \Lambda^n W$.

4. Proof of the theorem (dual version).

The proof relies on uniqueness of free resolutions.

First resolution: $(\tilde{T}^\cdot A, \tilde{d})$ - complex of free left A-modules

$$\tilde{T}^n A = A \otimes T^n A$$

(A-module structure = mult. on the ~~1st~~ 0^{th} tensor factor:

$$a \cdot (b \otimes \xi) = ab \otimes \xi \quad \forall\, a, b \in A,\ \xi \in T^n A)$$

$$\tilde{d} : \tilde{T}^n A \longrightarrow \tilde{T}^{n-1} A$$

$$a_0 \otimes \ldots \otimes a_n \longmapsto \sum_{i=0}^{n-1} (-1)^i \ldots a_i a_{i+1} \ldots$$

$$+ (-1)^n a_0 \otimes \ldots \otimes a_{n-1}\, \varepsilon(a_n)$$

$$\rightsquigarrow \quad \ldots \quad \tilde{T}^2(A) \xrightarrow{\tilde{d}} \tilde{T}^1(A) \xrightarrow{\tilde{d}} \tilde{T}^0(A) \xrightarrow{\varepsilon} \mathbb{C} \to 0 \quad - (4.1)$$

Claim: (4.1) is acyclic, hence a resolution of $\mathbb{C}$ viewed as A-module via $\varepsilon : A = S(W) \to \mathbb{C}$.

Proof. - Let $s : \tilde{T}^n(A) \longrightarrow \tilde{T}^{n+1}(A)$ be given by

$$a_0 \otimes \xi \longmapsto 1 \otimes a_0 \otimes \xi$$

$s : \mathbb{C} \to \tilde{T}^0(A) = A$ is $s(1) = 1$. One can verify easily that $\tilde{d} s + s \tilde{d} = Id$ which prove acyclicity of (4.1).

Second resolution (Koszul): $(K^\bullet A, \partial)$

$$K^n(A) = A \otimes \Lambda^n W \qquad (n \geq 0)$$

A-module str = again mult. on 0^{th} factor

$\partial : K^n(A) \longrightarrow K^{n-1}(A)$ is given by

$$\partial(a \otimes w_1 \wedge \dots \wedge w_n) = \sum_{i=1}^{n} (-1)^{i+1} a w_i \otimes (w_1 \wedge \dots \underset{\substack{\uparrow \\ \text{skipped.}}}{\widehat{w_i}} \wedge \dots w_n)$$

$$\rightsquigarrow \qquad \dots \xrightarrow{\partial} K^2(A) \xrightarrow{\partial} K^1(A) \xrightarrow{\partial} \underset{\substack{\| \\ A}}{K^0(A)} \to \mathbb{C} \quad - (4.2)$$

Claim: (4.2) is acyclic.

Proof Let $h : K^n(A) \longrightarrow K^{n+1}(A)$ be given by

$$h(w_1 \cdots w_m \otimes \xi) = \sum_{i=\phi}^{m} w_1 \cdots \hat{w}_i \cdots w_m \otimes w_i \wedge \xi$$

where $\begin{cases} w_1, \dots, w_m \in W \ ; \ w_1 \cdots w_m \in S^m(W) \subset A. \\ \xi \in \Lambda^n W \end{cases}$

Easy check: $(\partial h + h \partial)(P \otimes \xi) = (m+n) P \otimes \xi$

$\forall \ P \in S^m W, \ \xi \in \Lambda^n W.$

This proves acyclicity in degrees > 0. For $n = 0$:

$$K^1(A) = A \otimes W \xrightarrow{\partial} K^0(A) = A \quad \text{has cokernel} \cong A/_{\substack{\text{ideal} \\ \text{gen by} \\ W}} = \mathbb{C}.$$

$$a \otimes w \longmapsto aw$$

(4.2) is known as the <u>Koszul resolution</u>.

5. A straightforward computation shows that

$\tilde{\alpha} = Id_A \otimes \alpha : K^n(A) \longrightarrow \tilde{T}^n(A)$ is a chain map

(over $id : \mathbb{C} \to \mathbb{C}$).

That is

$$\begin{array}{ccccccccccc}
\cdots & K^n(A) & \xrightarrow{\partial} & K^{n-1}(A) & \to & \cdots & \to & K^0(A) & \to & \mathbb{C} \to 0 & (4.2) \\
 & \downarrow & & \downarrow & & \cdots & & \downarrow & & \| & \\
\cdots & \tilde{T}^n(A) & \xrightarrow[\tilde{d}]{} & \tilde{T}^{n-1}(A) & \to & \cdots & \to & \tilde{T}^0(A) & \to & \mathbb{C} \to 0 & (4.1)
\end{array}$$

all squares commute. By general results on uniqueness of free resolutions, $\exists\ \beta : \tilde{T}^n(A) \to K^n(A)$ s.t. $\tilde{\alpha} \circ \beta$ and $\beta \circ \tilde{\alpha}$ are homotopic to identity.

6. Final observation: $(T(A), d) = \mathbb{C} \otimes_A (\tilde{T}(A), \tilde{d})$

$(\Lambda^{\cdot}(W), 0) = \mathbb{C} \otimes_A (K(A), \partial)$

proves theorem ~~#~~ 3 and in turn theorem 2.

Remark. – This proof is due to Eilenberg-MacLane (1953). Cartier's proof (1957) uses similar arguments but in the setting of bicomplexes and spectral sequences.