0. Let (H, R) be a <u>quasitriangular Hopf algebra</u>. Let us recall the axioms:

(i) H is a <u>unital associative algebra</u> together with

(coproduct) • $\Delta : H \longrightarrow H \otimes H$ alg. hom. which is coassociative:

$$\Delta \otimes id\,(\Delta(x)) = id \otimes \Delta\,(\Delta(x)) \quad \forall\, x \in H.$$

(counit) • $\varepsilon : H \longrightarrow \mathbb{C}$ alg. hom. s.t.

$$\varepsilon \otimes id\,(\Delta(x)) = 1 \otimes x \in \mathbb{C} \otimes H$$
$$id \otimes \varepsilon\,(\Delta(x)) = x \otimes 1 \in H \otimes \mathbb{C}$$

(antipode) • $S : H \rightarrow H$ alg. anti hom $\big(S(xy) = S(y)\,S(x)\big)$ which is invertible and

$$mult \circ (S \otimes id) \circ \Delta = \varepsilon = mult \circ (id \otimes S) \circ \Delta.$$

(Note: $\Delta(S(x)) = S \otimes S\,(\Delta^{op}(x)) \quad \forall\, x \in H.$)

(ii) $R \in H \otimes H$ is an invertible element s.t.

$$\Delta^{op}(x) = R\,\Delta(x)\,R^{-1} \quad \forall\, x \in H. \quad (\text{intertwining eq}^{\underline{n}})$$

$$\Delta \otimes id\,(R) = R_{13}\,R_{23} \qquad (\text{cabling identities})$$
$$id \otimes \Delta\,(R) = R_{13}\,R_{12}$$

1. Sweedler's notation: Since Δ is coassociative, different ways of extending it to an alg. hom.

$$\Delta^{(n)} : H \longrightarrow H^{\otimes n}$$

are equal. That is, if we define $\Delta^{(n)}$ $(n \geq 0)$ as follows:

$\left(\Delta^{(0)} : H \to \mathbb{C}\right) =$ counit ; $\left(\Delta^{(1)} : H \to H\right) = id.$

$\left(\Delta^{(2)} : H \longrightarrow H \otimes H\right) = \Delta$; and for $n \geq 3$:

$$\Delta^{(n)} = \Delta \otimes id^{\otimes n-2} \circ \Delta^{(n-1)},$$

then $\Delta^{(n)}$ also equals $Id^{\otimes k} \otimes \Delta \otimes id^{\otimes (n-2-k)} \circ \Delta^{(n-1)}$.

Employing Sweedler's notation, we will write :

$$\Delta^{(n)}(x) = x^{(1)} \otimes \cdots \otimes x^{(n)}.$$

Thus, the axioms from the previous page take the following form:

<u>Counit</u> $\quad \varepsilon(x^{(1)}) \otimes x^{(2)} = 1 \otimes x$

$\quad x^{(1)} \otimes \varepsilon(x^{(2)}) = x \otimes 1$

<u>Antipode</u> $\quad S(x^{(1)}) \cdot x^{(2)} = \varepsilon(x) . 1 = x^{(1)} . S(x^{(2)})$

If $R = \sum_k a_k \otimes b_k$, then :

<u>Intertwining eq^n</u> : $\quad a_k x^{(1)} \otimes b_k x^{(2)} = x^{(2)} a_k \otimes x^{(1)} b_k$

<u>Cabling id-s</u> $\quad a_k^{(1)} \otimes a_k^{(2)} \otimes b_k = a_k \otimes a_\ell \otimes b_k b_\ell$

$\quad a_k \otimes b_k^{(1)} \otimes b_k^{(2)} = a_k a_\ell \otimes b_\ell \otimes b_k$

Ex : Show that, $\forall n \geq 2$:

$$\Delta^{(n)} \otimes id \, (R) = R_{1,n+1} R_{2,n+1} \cdots R_{n,n+1} \quad \left(\text{in } H^{\otimes(n+1)}\right).$$

2. Some identities following directly from the axioms :

Prop. (1) $\varepsilon \otimes id\,(R) = 1 \otimes 1 = id \otimes \varepsilon\,(R)$

(2) $S \otimes id\,(R) = R^{-1} = id \otimes S^{-1}\,(R)$

Proof (1) Start from $\Delta \otimes id\,(R) = R_{13} \cdot R_{23}$; apply $\varepsilon \otimes id \otimes id$ and multiply 1^{st} and 2^{nd} tensors :

$$a_k^{(1)} \otimes a_k^{(2)} \otimes b_k = a_k \otimes a_\ell \otimes b_k b_\ell$$

$$\rightsquigarrow \quad \underbrace{\varepsilon(a_k^{(1)})\, a_k^{(2)}}_{\text{counit axiom}} \otimes b_k = \varepsilon(a_k)\, a_\ell \otimes b_k b_\ell$$

$$\Rightarrow \quad R = \varepsilon \otimes id\,(R) \cdot R \quad \Rightarrow \quad \varepsilon \otimes id\,(R) = 1 \otimes 1 .$$

Proof of $id \otimes \varepsilon\,(R) = 1 \otimes 1$ is similar.

(2): Again $\Delta \otimes id\,(R) = R_{13} \cdot R_{23}$ ← apply $S \otimes id \otimes id$ and multiply 1^{st} & 2^{nd} tensor factors

$$\varepsilon \otimes id\,(R) = S \otimes id\,(R) \cdot R$$

$$\overset{\displaystyle \|}{1 \otimes 1} \qquad \Rightarrow \quad S \otimes id\,(R) = R^{-1} .$$

□

3. <u>Drinfeld element</u> <u>$u \in H$</u> is defined to be :

$$u = mult \circ S \otimes id\,(R_{21})$$

$$= S(b_k)\, a_k$$

Proposition. (i) For every $x \in H$, $[\Delta(x), R_{21}R] = 0$.

(ii) $S^2(x)\cdot u = u \cdot x$, $\forall x \in H$.

(iii) $u \in H$ is invertible with $v = u^{-1} = b_k\, S^2(a_k)$
(i.e., $v = (\text{mult} \circ id \otimes S^2)(R_{21})$.)

(iv) $\Delta(u)\cdot R_{21}R = u \otimes u$.

Proof. (i) Since $\Delta^{op}(x)\cdot R = R \cdot \Delta(x)$, we get by applying (12):

$$\Delta(x)\cdot R_{21} = R_{21}\cdot \Delta^{op}(x) = R_{21}\cdot R\,\Delta(x)\,R^{-1}$$

$$\Rightarrow \quad \Delta(x)\cdot R_{21}R = R_{21}R\cdot\Delta(x). \quad \square$$

(ii) Consider the relation $R_{12}\cdot\Delta^{(3)}(x) = \Delta^{op}\otimes id\,(\Delta(x))\cdot R_{12}$.

Written in Sweedler's notation, with $R = a_k \otimes b_k$:

$$a_k x^{(1)} \otimes b_k x^{(2)} \otimes x^{(3)} = x^{(2)} a_k \otimes x^{(1)} b_k \otimes x^{(3)}.$$

Apply $id \otimes S \otimes S^2$ to this identity, and multiply factors $3^{rd}\cdot 2^{nd}\cdot 1^{st}$:

$$(*):\quad S^2(x^{(3)})\cdot S(x^{(2)})\,S(b_k)\cdot a_k x^{(1)} = S^2(x^{(3)})\cdot S(b_k)\,S(x^{(1)})\cdot x^{(2)} a_k$$

$$\text{L.H.S. of } (*) = S\left(x^{(2)}\cdot S(x^{(3)})\right)\cdot u \cdot x^{(1)}$$

$$= u\cdot x$$

(antipode and counit axiom: $x^{(1)}\otimes x^{(2)}\cdot S(x^{(3)}) = id\otimes\varepsilon\,(\Delta(x)) = x \otimes 1$.)

$$\text{R.H.S. of } (*) = S^2(x^{(3)})\cdot S(b_k)\,\underbrace{\boxed{S(x^{(1)})\cdot x^{(2)}}}_{\text{antipode axiom}}\,a_k$$

$$= S^2(x)\cdot u \quad \text{as claimed.} \quad \square$$

(iii) $$vu = b_k S^2(a_k)\cdot u = b_k \cdot u \cdot a_k \quad (\text{by (ii)})$$
$$= b_k S(b_l)\, a_l\, a_k \quad (\text{defn. of } u)$$
$$= S(b_k)\, S(b_l)\, a_l\, S(a_k) \quad (\text{since } S\otimes S(R) = R \text{ by Prop. 2 above})$$

Now $S \otimes id(R) = R^{-1}$. Let us write $R^{-1} = \sum a_k' \otimes b_k'$. Then:

$$vu = S(b_k')\, S(b_l)\, a_l\, a_k' = 1 \quad \text{since } a_l a_k' \otimes b_l b_k' = R\cdot R^{-1} = 1\otimes 1.$$

Proof of $u\cdot v = 1$ is similar.

(iv) $$\Delta(u)\cdot R_{21}R = \Delta(S(b_k))\cdot \Delta(a_k)\cdot R_{21}R$$
$$= S\otimes S(\Delta^{op}(b_k))\cdot R_{21}R\cdot \Delta(a_k) \quad \left(\begin{array}{l}\text{using (i)}\\ \text{above}\end{array}\right).$$

Using cabling identities:

$$\Delta(a_k) \otimes \Delta^{op}(b_k) = \Delta \otimes \Delta^{op}(R)$$
$$= R_{13} R_{14} R_{23} R_{24} = R_{13} R_{23} R_{14} R_{24} \in H^{\otimes 4}.$$

To write $\Delta(u) R_{21} R$ more compactly, we use the following notation:

$$(H\otimes H) \circlearrowleft H^{\otimes 4} : \quad X \triangleleft (A \otimes B) = S\otimes S(B)\cdot X \cdot A$$
$$(\forall\ A, B, X \in H\otimes H)$$

Hence $$\Delta(u) R_{21} R = R_{21} \triangleleft (\underbrace{R_{12} R_{13} R_{23}}_{\text{Yang-Baxter eq}^n} R_{14} R_{24})$$
$$= R_{21} \triangleleft (R_{23} R_{13} R_{12} R_{14} R_{24})$$

we will compute this one term at a time :

$$R_{21} \triangleleft R_{23} = (S(b_\ell)\otimes 1)\cdot(b_k \otimes a_k)\cdot(1\otimes a_\ell)$$

$$= S(b_\ell)\, b_k \otimes a_k a_\ell$$

$$= S\otimes id\left(S^{-1}\otimes id\,(R_{21})\cdot R_{21}\right) = 1\otimes 1$$

because $id \otimes S^{-1}(R) = R^{-1}$
by Prop. 2 above.

$$1\otimes 1 \triangleleft R_{13} = S(b_\ell)\otimes 1 \cdot 1\otimes 1 \cdot a_\ell \otimes 1 = S(b_\ell)a_\ell \otimes 1$$

$$= u \otimes 1$$

$$u\otimes 1 \triangleleft R_{12} = (u\otimes 1)R = u a_k \otimes b_k$$

$$(u\otimes 1)R \triangleleft R_{14} = (1\otimes S(b_\ell))\,(u a_k \otimes b_k)\,(a_\ell \otimes 1)$$

$$= u\, a_k a_\ell \otimes S(b_\ell)\, b_k = u\otimes 1$$ because

$$u\otimes 1 \triangleleft R_{24} = 1\otimes S(b_\ell)\cdot u\otimes 1 \cdot 1\otimes a_\ell$$

$$= u\otimes u$$

Hence : $\Delta(u)\, R_{21} R = u\otimes u.$ $\square$

4. $\underline{H = U_\hbar(\mathfrak{g})}$. Recall the formulae for Δ and S on the generators $\{E_i, F_i\}_{i\in I}$

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \qquad S(E_i) = -K_i^{-1}E_i$$

$$\Delta(F_i) = F_i \otimes K_i^{-1} + 1\otimes F_i \qquad S(F_i) = -F_i K_i$$

$\Delta(K) = K\otimes K$
$S(K) = K^{-1}$.
for any $K = q^h$
$h \in \mathfrak{h}$.

$$\Rightarrow \quad S^2(E_i) = -S(E_i)\,S(K_i^{-1}) = K_i^{-1} E_i K_i \left(= q_i^{-2} E_i\right)$$

$$S^2(F_i) = -S(K_i)\,S(F_i) = K_i^{-1} F_i K_i \left(= q_i^{2} F_i\right) \quad \forall i \in I.$$

Therefore, if we set $K_{2\rho} = q^{2\nu(\rho)}$ where $\rho \in \mathfrak{h}^* \underset{\nu}{\xrightarrow{\sim}} \mathfrak{h}$ is such that $\rho(h_i) = 1$ $(\forall i \in I)$; then

$$\boxed{S^2(x) = K_{2\rho}^{-1}\, x\, K_{2\rho} \quad \forall x \in U_\hbar(\mathfrak{g})}$$

Let $u \in U_\hbar(\mathfrak{g})$ be the Drinfeld element. By Prop. 3 (ii),

$$S^2(x) = u \cdot x \cdot u^{-1} \quad \forall x \in U_\hbar(\mathfrak{g}).$$

Thus, we obtain: $\boxed{\theta = K_{2\rho}^{-1} \cdot u^{-1}}$ is central in $U_\hbar(\mathfrak{g})$

Proposition: (1) $R_{21} \cdot R = \Delta(\theta) \cdot \theta^{-1} \otimes \theta^{-1}$

(2) For an iso. $\underset{(alg)}{\psi}$: $U_\hbar(\mathfrak{g}) \to U(\mathfrak{g})[[\hbar]]$;

$\psi(\theta) = e^{\frac{\hbar}{2} C}$ where $C \in U(\mathfrak{g})$ is the Casimir element.

Proof. (1) Using Prop 3 (iv): $\Delta(u)\, R_{21} R = u \otimes u$

$$\Rightarrow \quad R_{21} \cdot R = \Delta(u^{-1}) \cdot u \otimes u$$

$$= \Delta(K_{2\rho}\theta) \cdot K_{2\rho}^{-1} \otimes K_{2\rho}^{-1} \cdot \theta^{-1} \otimes \theta^{-1}$$

$$= \Delta(\theta) \cdot \theta^{-1} \otimes \theta^{-1}. \quad \text{since } \Delta(K_{2\rho}) = K_{2\rho} \otimes K_{2\rho}.$$

(2) Recall: $C = \underset{\substack{\uparrow \\ \text{Cartan part}}}{C_0} + \sum_{\alpha \in R_+} x_\alpha^+ x_\alpha^- + x_\alpha^- x_\alpha^+$

$$= C_0 - 2\,v(\rho) + 2 \sum_{\alpha \in R_+} x_\alpha^+ x_\alpha^-$$

$\Rightarrow$ C acts on a lowest weight vector of weight $\mu \in \mathfrak{h}^*$ as:

$$C = (\mu, \mu) - (\mu, 2\rho)$$

(i.e. if $\mathcal{U}(\mathfrak{g}) \circlearrowright V$ and $v \in V[\mu]$ is such that $f_i . v = 0 \ \forall i \in I$

then $C.v = (\mu, \mu - 2\rho)\, v$.

Since C is central; if this $v \in V[\mu]$ generates the entire V as a $\mathcal{U}(\mathfrak{g})$-module, then $C|_V = (\mu, \mu - 2\rho). \mathrm{Id}_V$.)

$HH^1(\mathcal{U}(\mathfrak{g}); \mathrm{End}\, V) = 0 \Rightarrow V$ cannot be non-trivially deformed.

Considering the lowest weight repn. of $\mathcal{U}_\hbar(\mathfrak{g})$, and

$$R = q^{\Omega_0} . \sum_\gamma F_\gamma \otimes E_\gamma$$

we get that $u = \mathrm{mult} \circ \cancel{id\otimes}\, S \otimes id\, (R_{21})$ acts on $v \in V[\mu]$ lowest wt. vector

by $q^{-(\mu,\mu)}$.

$\Rightarrow$ θ acts on v as: $\quad \theta = K_{2\rho}^{-1} . u^{-1} : v \mapsto q^{(\mu,\mu) - (\mu, 2\rho)} . v$

i.e. $\theta = q^{C}$ (on f.d. repns.) $\square$