Lecture 4.

PDE's with regular singularities along hyperplanes.

0. Let V be an n-dimensional $\mathbb{C}$-vector space. ($n \in \mathbb{Z}_{\geq 1}$).

Let $\mathcal{U}$ be a unital associative $\mathbb{C}$-algebra (e.g. $M_{N\times N}(\mathbb{C})$).

Consider a system of differential equations; for a $\mathcal{U}$-valued function on V; $F : D \to \mathcal{U}$ ($D \subset V$ open disc, around $\underline{0} \in V$.)

$$\frac{\partial F}{\partial z_j} = \mathcal{A}_j(z) F(z) \quad - \quad (*)$$

Here we pick a basis of the dual vector space V^*, denoted by $\{z_1, \ldots, z_{\cancel{\#}n}\}$, which allows us to view F as a function of ~~#~~ n-complex variables. Each $\mathcal{A}_j(z)$ is a meromorphic function, also $\mathcal{U}$-valued.

1. Consistency condition. The system of PDE's, $(*)$ is said to be consistent (or integrable, or flat) if $\forall\, j,k \in \{1,\ldots,n\}$; $j \neq k$;

$$(1.1) - \quad \frac{\partial}{\partial z_j}\mathcal{A}_k(z) - \frac{\partial}{\partial z_k}\mathcal{A}_j(z) - [\mathcal{A}_j(z), \mathcal{A}_k(z)] = 0$$

Lemma. (Assume 0 is a regular point). $(*)$ admits a unique solution of the form $F(0) = 1$, iff (1.1) holds. (i.e. the system is consistent). $\boxed{\text{or any invertible element of } \mathcal{U}}$

Proof. Assume a solution (invertible) exists, say $G(\underline{z})$.

Then $(\forall j \neq k)$:
$$\frac{\partial^2 G}{\partial z_k \partial z_j} = \frac{\partial}{\partial z_k}(A_j \cdot G)$$
$$= \left(\frac{\partial A_j}{\partial z_k}\right) \cdot G + A_j \cdot A_k \cdot G$$

and
$$\frac{\partial^2 G}{\partial z_j \partial z_k} = \left(\frac{\partial A_k}{\partial z_j}\right) G + A_k \cdot A_j \cdot G.$$

As G is invertible, $\partial_j A_k - \partial_k A_j - [A_j, A_k] = 0$

Hence (1.1) is necessary for such a G to exist.

Let us prove the sufficiency for $n = 2$ case.

(Notation: $\partial_1 = \frac{\partial}{\partial z_1}$, $\partial_2 = \frac{\partial}{\partial z_2}$). Our equations are:

$$\boxed{\partial_1 F = A_1 \cdot F} \quad \text{and} \quad \partial_2 F = A_2 \cdot F.$$

Fix z_2 and view it as a one variable system near ordinary point $z_1 = 0$. Our solution then necessarily be a holomorphic function of z_2. Let us denote this solution by $\psi(z_1, z_2)$.

(see the proof of Prop. 2, Lecture 1) or, any textbook on ODE's.

Our task now, is to find a function $C(z_2)$ so that $\psi(z_1,z_2)\cdot C(z_2)$ solves the second eq^n $\partial_2 F = A_2 \cdot F$.

Since $C(z_2)$ is a "constant" for ∂_1; we will thus obtain $G = \psi \cdot C$ a joint solution.

$\psi(z_1,z_2)\cdot C(z_2)$ solves $\partial_2 F = A_2 F$ $\iff$ $C'(z_2) = \psi(z_1,z_2)^{-1}\left(\partial_{z_2}\psi - A_2\psi\right)\cdot C$ (easy exercise)

Thus, we will be done by our one-variable argument, provided we check that $\psi^{-1}(\partial_{z_2}\psi - A_2\cdot\psi)$ is independent of z_1. That is,

Such a $C(z_2)$ exists (& is unique provided $C(0)$ is fixed) $\iff$ $\partial_{z_1}\left(\psi^{-1}\cdot(\partial_2\psi - A_2\psi)\right) = 0$

$$\iff -\psi^{-1}A_1(\partial_2\psi - A_2\psi) + \psi^{-1}\left(\partial_2(\partial_1\psi) - \partial_1 A_2\cdot\psi - A_2A_1\psi\right) = 0$$

$$\iff \left(\partial_2 A_1 - \partial_1 A_2 + [A_1, A_2]\right)\cdot\psi = 0$$

The general case follows an easy induction argument.

2. Language of differential forms.

- Let us denote by $\Gamma(D;\mathcal{U}) = \{\mathcal{U}\text{-valued (holomorphic) functions on } D\}$

- $\Omega^1(D;\mathcal{U})$ = $\mathcal{U}$-valued 1-forms on D.

  a typical element is of the form $\sum_{j=1}^{N} f_j(\underline{z})\, dz_j$

i.e. $\Omega^1(D;\mathcal{U}) = \bigoplus_{j=1}^{\cancel{N}n} \Gamma(D;\mathcal{U})\cdot dz_j$

- de Rham differential $d : \Gamma(D;\mathcal{U}) \to \Omega^1(D;\mathcal{U})$

$$df = \sum_{j=1}^{\cancel{N}n} \frac{\partial f}{\partial z_j}\, dz_j$$

Our system of PDE's can then be written more compactly as $\nabla \cdot F = 0$; when $\nabla = d - \mathcal{A}$, and we view $\mathcal{A} = \sum_{j=1}^{\cancel{N}n} \mathcal{A}_j(\underline{z})\, dz_j \in \Omega^1(D;\mathcal{U})$. The consistency equation can be written as an identity in $\Omega^2(D;\mathcal{U})$

- $\Omega^2(D;\mathcal{U}) = \bigoplus_{j<k} \Gamma(D;\mathcal{U})\, dz_j \wedge dz_k$ (rank $\binom{n}{2}$ module over $\Gamma(D;\mathcal{U})$)

  [commutation rule: $dz_{j_2} \wedge dz_{j_1} = -dz_{j_1} \wedge dz_{j_2}$]

Lemma. The system of PDE's $\nabla F = 0$ ($\equiv$ly $\partial_j F = \mathcal{A}_j(\underline{z}) \cdot F(\underline{z})$) is consistent $\Longleftrightarrow$ $d\mathcal{A} - \mathcal{A} \wedge \mathcal{A} = 0$

(here $\mathcal{A} = \sum_{j=1}^{N} \mathcal{A}_j\, dz_j$).

Proof. $d\mathcal{A} = d\left(\sum_{j=1}^{n} \mathcal{A}_j \, dz_j\right)$

$$= \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \frac{\partial \mathcal{A}_j}{\partial z_k} dz_k \right) \wedge dz_j$$

Coeff of $dz_{j_1} \wedge dz_{j_2}$ is $\dfrac{\partial \mathcal{A}_{j_2}}{\partial z_{j_1}} - \dfrac{\partial \mathcal{A}_{j_1}}{\partial z_{j_2}}$.
$(j_1 < j_2)$

Similarly $\mathcal{A} \wedge \mathcal{A} = \left(\sum_{j=1}^{n} \mathcal{A}_j \, dz_j\right) \wedge \left(\sum_{k=1}^{n} \mathcal{A}_k \, dz_k\right)$

and coefficient of $dz_{j_1} \wedge dz_{j_2}$ here is: $\mathcal{A}_{j_1} \cdot \mathcal{A}_{j_2} - \mathcal{A}_{j_2} \mathcal{A}_{j_1}$
$(j_1 < j_2)$

Hence $d\mathcal{A} - \mathcal{A} \wedge \mathcal{A} = \sum_{j_1 < j_2} \left(\partial_{j_1} \mathcal{A}_{j_2} - \partial_{j_2} \mathcal{A}_{j_1} - [\mathcal{A}_{j_1}, \mathcal{A}_{j_2}] \right) dz_{j_1} \wedge dz_{j_2}$

and the lemma follows. □

3. Hyperplane singularities (regular). We will consider the following special type of PDE's :

Let $X \subset V^* \setminus \{0\}$ be a finite set of linear forms on V. Assume given $t_x \in \mathcal{U}$ $\forall x \in X$.

Also (WLOG) we assume that $x \neq y$, $x, y \in X$ $\Rightarrow$ x and y are not proportional.

$$\nabla := d - \sum_{x \in X} \frac{dx}{x} \cdot t_x$$

More explicitly, if $x_1, \dots, x_n \in V^*$ is a basis of V^*, then our PDE, namely $\nabla F = 0$, takes the following form.

$$\frac{\partial F}{\partial x_j} = \sum_{x \in X} \frac{n_j(x)}{x} \, t_x \cdot F \; ; \quad \text{where } x = \sum_{i=1}^{n} n_i(x) \, x_i$$

$(\in \mathbb{C})$

Lemma (Kohno) $\nabla = d - \sum_{x \in X} \frac{dx}{x} t_x$ is flat if and only if: $\forall$ 2-dimensional subspace $V_1 \subset V^*$; $\sum_{x \in X \cap V_1} t_x$ commutes with t_y $\forall y \in X \cap V_1$.

[Note: it is enough to take V_1 = span of two distinct elements of X.]

Proof.- As $\nabla = d - \mathcal{A}$, we need to prove the equivalence of the stated commutation relations and

$$d\mathcal{A} - \mathcal{A} \wedge \mathcal{A} = 0. \qquad \text{Here } \mathcal{A} = \sum_{x \in X} \frac{dx}{x} \cdot t_x .$$

An easy computation shows that $d\mathcal{A} = 0$.

Now, $\mathcal{A} \wedge \mathcal{A} = \frac{1}{2} \sum_{x,y \in X} \frac{dx}{x} \wedge \frac{dy}{y} [t_x, t_y]$.

$$\mathcal{A} \wedge \mathcal{A} = 0 \underset{(\Leftarrow)}{\Rightarrow} x \cdot (\mathcal{A} \wedge \mathcal{A})\Big|_{x=0} = 0 \quad \forall x \in X.$$

(With some work it can be shown to be equivalent, - left as an exercise.)

Now $\quad x \cdot \mathcal{A} \wedge \mathcal{A}\Big|_{x=0} = dx \wedge \sum_{y \in X} \frac{dy}{y}\Big|_{x=0} [t_x, t_y]$

Let us define an exquivalence relation on $X \setminus \{x\}$ as:

$y \sim y' \Leftrightarrow y\big|_{x=0}$ is proportional to $y'\big|_{x=0}$; i.e. $y' \in \operatorname{Span}\{x, y\}$ & $y \in \operatorname{Span}\{x, y'\}$

$X_1 \sqcup \dots \sqcup X_K$ = decomposition of $X \setminus \{x\}$ into equivalence classes.

$y_j \in X_j$ a choice from each equivalence class.

Then $\quad x \cdot \mathcal{A} \wedge \mathcal{A}\Big|_{x=0} = \sum_{j=1}^{K} dx \wedge \frac{dy_j}{y_j} [t_x, \sum_{y \in X_j} t_y]$

$= 0 \Leftrightarrow \forall j, \quad [t_x, \sum_{y \in X_j} t_y] = 0$. Note that this condition is equivalent to the one stated above. $\square$

4. Example Assume $n=2$, V is 2-dim'l

x, y basis of V^*.

$$X = \{x, y, x+y\}$$

$$\nabla = d - \left(\frac{dx}{x} \cdot t_1 + \frac{dy}{y} t_2 + \frac{d(x+y)}{x+y} t_3 \right)$$

$$= d - \left(dx \cdot \left(\frac{t_1}{x} + \frac{t_3}{x+y} \right) + dy \left(\frac{t_2}{y} + \frac{t_3}{x+y} \right) \right)$$

PDE: $\dfrac{\partial f}{\partial x} = \left(\dfrac{t_1}{x} + \dfrac{t_3}{x+y} \right) \cdot f$; $\dfrac{\partial f}{\partial y} = \left(\dfrac{t_2}{y} + \dfrac{t_3}{x+y} \right) \cdot f$

Equation (1.1) – or its equivalent form in our case, given by Kohno's lemma's proof : $\mathcal{A} \wedge \mathcal{A} = 0$

$$\iff dx \wedge dy \cdot \left(\frac{[t_1, t_2]}{xy} + \frac{[t_1, t_3]}{x(x+y)} - \frac{[t_2, t_3]}{y(x+y)} \right) = 0$$

i.e. (clear denominator) $(x+y)[t_1, t_2] + y[t_1, t_3] - x[t_2, t_3] = 0$

i.e. $x([t_1 + t_3, t_2]) + y[t_1, t_2 + t_3] = 0$

In other words $t_1 + t_2 + t_3$ commutes with t_1, t_2 & t_3.