Lecture 5

Hyperplane complements/arrangements.

0. Recall that last time we started studying PDE's of the form. $\nabla F = 0$; where $\nabla = d - \sum_{x \in X} \frac{dx}{x} t_x$.

Its domain of definition is $V^{reg} = V \setminus \bigcup_{x \in X} Ker(x)$.

Recall:
- V is n-dimensional $\mathbb{C}$-vector space.
- $X \subset V^* = \mathrm{Hom}_{\mathbb{C}\text{-linear}}(V, \mathbb{C})$. Assumptions on X:
  - (1) $0 \notin X$.
  - (2) $x, y \in X$; $x \neq y \Rightarrow x$ and y are not proportional
  - (3) Span of $X = V^*$.
- $\forall x \in X$, $H_x = Ker(x) \subset V$ is a hyperplane.

The system of PDE's $\nabla F = 0$, where $\nabla = d - \sum_{x \in X} \frac{dx}{x} t_x$, for $F : V^{reg} \to \mathcal{U}$ (a unital associative algebra over $\mathbb{C}$, such as $M_{N \times N}(\mathbb{C})$).

has regular singularities along H_x $(x \in X)$. It is consistent if and only if for every maximal $Y \subset X$ s.t. $\dim(\text{Span of } Y) = 2$

(Kohno's lemma - lecture 4, §3, p.6)

we have: $\left[\sum_{y\in Y} t_y , t_{y'}\right] = 0 \quad \forall y' \in Y.$

1. Remarks. - The <u>holonomy Lie algebra</u> of an arrangement of hyperplanes $\{H_x\}_{x\in X}$; is defined as the free algebra generated by $\{t_x\}_{x\in X}$ subject to the relations

$\forall\, Y \subset X$ max'l s.t. Span(Y) is 2-dim'l ; $\sum_{y\in Y} t_y$ commutes with t_z $\forall z \in Y$.

In simple terms, in order to write down a consistent system of PDE's as above, we need to find matrices $t_x \in M_{N\times N}(\mathbb{C})$ which obey the "rank 2 relations" as above.

e.g. $V = \mathbb{C}^n$, $X = \{z_i - z_j : 1 \le i < j \le n\}$

$V^{reg} = \mathbb{C}^n \setminus \{\underline{z} \mid z_i = z_j \text{ for some } i \ne j\}$ = configuration space of n ordered points on $\mathbb{C}$.

Holonomy relations: $\{t_{ij} \mid 1 \le i < j \le n\}$

$$\begin{cases} [t_{ij}, t_{kl}] = 0 & \text{from } Y = \{z_i - z_j, z_k - z_l\} \quad i,j,k,l \text{ distinct} \\ [t_{ij}, t_{jk} + t_{ik}] = 0 & Y = \{z_i - z_j, z_j - z_k, z_i - z_k\} \quad i,j,k \text{ distinct} \end{cases}$$

2. Root systems – examples of hyperplane arrangements.

Let E be a finite-dimensional real vector space; together with a positive definite symmetric bilinear form $(\cdot\,,\cdot): E\times E \to \mathbb{R}$.

We identify $\nu: E^* \longrightarrow E$ by the rule

$$(\nu(\alpha), v) = \alpha(v) \quad \forall \begin{array}{l}\alpha\in E^*\\ v\in E.\end{array}$$

and use $(\cdot\,,\cdot): E^*\times E^* \to \mathbb{R}$ for the induced inner product.

$$(\alpha,\beta) = (\nu(\alpha), \nu(\beta))$$

Definition For every $\alpha\in E^*$, $\alpha\neq 0$, let $\alpha^\vee = \dfrac{2.\nu(\alpha)}{(\alpha,\alpha)} \in E$.

$$S_\alpha : E \longrightarrow E \qquad \left(\text{and } S_\alpha : E^* \longrightarrow E^* \atop \gamma \longmapsto \gamma - \gamma(\alpha^\vee).\alpha\right)$$

$$v \longmapsto v - \alpha(v).\alpha^\vee$$

is a reflection through $H_\alpha = \mathrm{Ker}(\alpha)$.

(i.e. $S_\alpha(v) = v \;\; \forall\, v\in H_\alpha$

$S_\alpha^2 = \mathrm{Id}_E$. In fact $S_\alpha(\alpha^\vee) = -\alpha^\vee$).

3. A <u>root system</u> is a finite set $R \subset E^* \setminus \{0\}$ s.t.

(1) R spans E^*.

(2) $\alpha,\beta\in R$ and $\alpha = c\beta \Rightarrow c = \pm 1$.

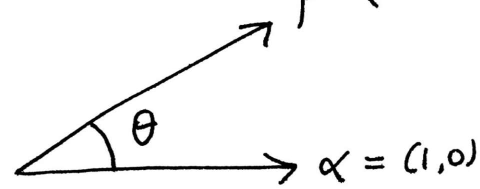
(3) $\forall\, \alpha,\beta \in \mathbb{Z}R$; $\beta(\alpha^\vee) = \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.

(4) $\forall\, \alpha\in \mathbb{Z}R$: $S_\alpha : E^*\to E^*$ preserves R.

4. Classification of rank 2 root systems.

Assume E is 2 dimensional and let $R \subset E^* \setminus \{0\}$ be a root system. Pick $\alpha, \beta \in R$ non-proportional.

Upon rescaling (all the elements of R – so that the stated conditions remain true) and using $S_\alpha(R) = R$, we may assume that $|\alpha| = 1$, $|\beta| \geq 1$ and the angle between α & β is acute.



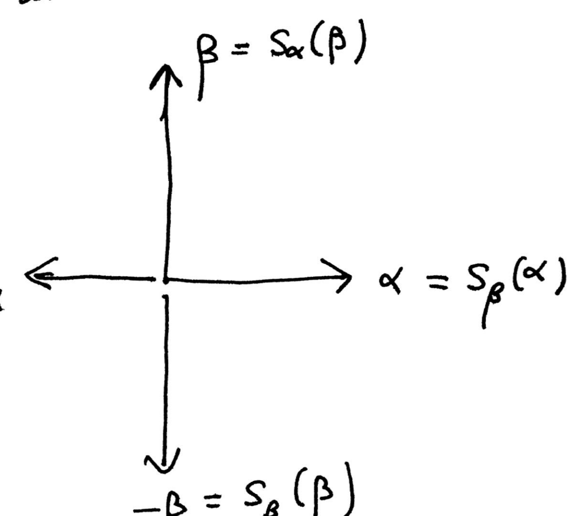
(We have identified, thus, $E \simeq \mathbb{R}^2$ with its standard inner product – $(v,w) = |v| \cdot |w| \cos(\angle vw)$).

Integrality constraint:
$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = 2r\cos(\theta) \in \mathbb{Z}$$
$$\frac{2(\beta,\alpha)}{(\beta,\beta)} = \frac{2}{r}\cos(\theta) \in \mathbb{Z}$$

$\Rightarrow 4\cos^2(\theta) \in \mathbb{Z}$ and hence $4\cos^2(\theta) = 0, 1, 2, 3$ or 4.

$(A_1 \times A_1)$ $\quad 4\cos^2(\theta) = 0$, i.e. $\theta = \frac{\pi}{2}$ and no condition on r

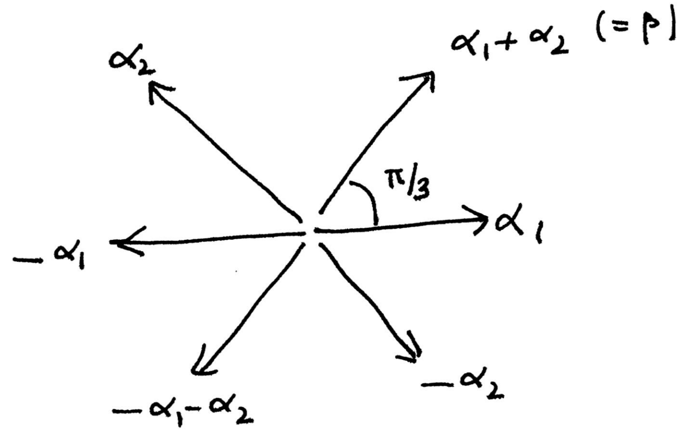
Reducible root system $\leftrightarrow$



(A_2). $4\cos^2(\theta) = 1 \Rightarrow \cos(\theta) = \frac{1}{2}$ i.e. $\theta = \frac{\pi}{3}$.

Integrality, $\frac{2}{r} \cdot \frac{1}{2} \in \mathbb{Z} \Rightarrow r = 1$. So α and β are of the same length.

Root system of type A_2:

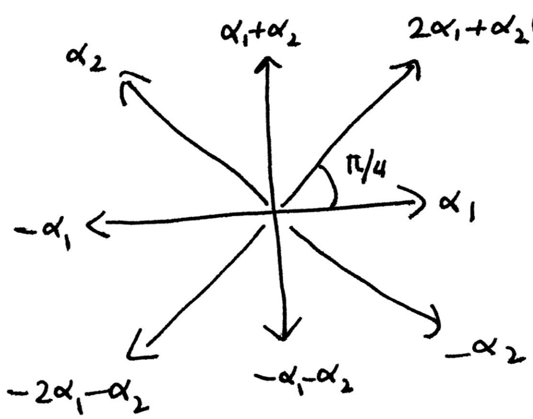


(B_2) $4\cos^2(\theta) = 2 \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$ i.e. $\theta = \frac{\pi}{4}$.

Again by integrality

$$\left.\begin{array}{l} 2r \cdot \frac{1}{\sqrt{2}} \in \mathbb{Z} \\ \\ \frac{2}{r} \cdot \frac{1}{\sqrt{2}} \in \mathbb{Z} \end{array}\right\} \Rightarrow r = \sqrt{2}.$$

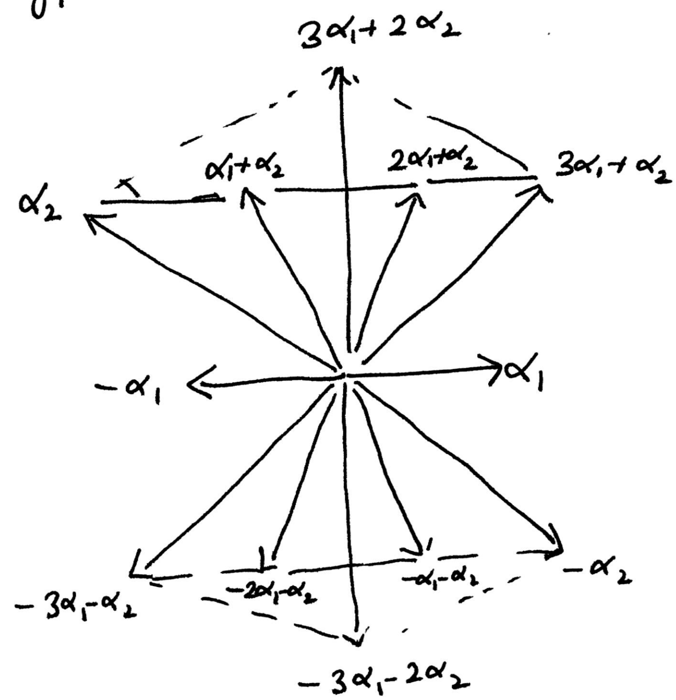
$|\beta|^2 = 2$. Root system of type B_2:



(G_2). $\quad 4\cos^2(\theta) = 3 \;\Rightarrow\; \cos(\theta) = \frac{\sqrt{3}}{2} \quad$ i.e. $\theta = \frac{\pi}{6}$.

Integrality:
$$\left.\begin{array}{l} 2r \cdot \frac{\sqrt{3}}{2} \in \mathbb{Z} \\ \frac{2}{r} \cdot \frac{\sqrt{3}}{2} \in \mathbb{Z} \end{array}\right\} \Rightarrow r = \sqrt{3}$$
$$(|\beta|^2 = 3)$$

Picture of the root system of type G_2.



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5. More definitions related to root systems.

- W = subgroup of $GL(E)$ (or $GL(E^*)$) generated by $\{s_\alpha\}_{\alpha \in R}$. called <u>Weyl group</u> of the root system.

Note: since s_α preserve R and R spans E^*,

Later

$W <$ Permutation group on R and hence finite.