Lecture 6 - Root Systems*

0. Recall that we defined a root system as follows:

Let E be a finite-dimensional real vector space, together with a positive-definite form $(\cdot,\cdot) : E \times E \to \mathbb{R}$. Using $(\cdot,\cdot)$ we identified $\nu : E^* \xrightarrow{\sim} E$ according to the rule $(\nu(\alpha), \phi) = \alpha(\phi)$ and $(\forall \alpha \in E^*, \phi \in E)$

transported $(\cdot,\cdot)$ on E^* (i.e. $(\alpha, \beta) \underset{def}{=} (\nu(\alpha), \nu(\beta)) \ \forall \alpha, \beta \in E^*$).

Notation: $\forall \alpha \in E^*$, $\alpha \neq 0$, we define $\alpha^\vee \in E$ by:

$$\alpha^\vee = \frac{2\,\nu(\alpha)}{(\alpha,\alpha)} .$$

(Note: $\alpha(\alpha^\vee) = 2 \ \forall \alpha \in E^* \setminus \{0\}$.)

<u>Reflections</u>: Let $\alpha \in E^*$; $\alpha \neq 0$. We have $s_\alpha \in GL(E)$ or $GL(E^*)$ given by

$$s_\alpha : E \to E, \quad \phi \mapsto \phi - \alpha(\phi)\cdot\alpha^\vee \quad ; \quad s_\alpha : E^* \to E^*, \quad \gamma \mapsto \gamma - \gamma(\alpha^\vee)\alpha$$

By a <u>root system</u> we mean a finite set $R \subset E^* \setminus \{0\}$ s.t.

(1) R spans E^*.

(2) $\alpha, c\alpha \in R \Rightarrow c = \pm 1$.

(3) $\alpha, \beta \in R \Rightarrow \beta(\alpha^\vee) = \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

(4) $\forall \alpha \in R$; $s_\alpha(R) \subset R$.

In two-dimensions, we obtained the following classification:

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* cf - Bourbaki - Lie groups & Lie algebras Ch 4, 5, 6.

$(A_1 \times A_1)$ $\quad R = \{\pm\alpha, \pm\beta\}$, $(\alpha, \beta) = 0$.

(A_2) $\quad R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$; $\quad |\alpha_1|^2 = |\alpha_2|^2$

and $\alpha_1(\alpha_2^\vee) = \alpha_2(\alpha_1^\vee) = -1$.

(B_2) $\quad R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2)\}$;

$|\alpha_2|^2 = 2|\alpha_1|^2$ and $\alpha_1(\alpha_2^\vee) = -1$

$\alpha_2(\alpha_1^\vee) = -2$

(G_2) $\quad R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2),$

$\pm(3\alpha_1+\alpha_2), \pm(3\alpha_1+2\alpha_2)\}$;

$|\alpha_2|^2 = 3|\alpha_1|^2$ and $\alpha_1(\alpha_2^\vee) = -1$

$\alpha_2(\alpha_1^\vee) = -3$

1. Positive and negative roots.

For $\alpha \in E^* \setminus \{0\}$, let $H_\alpha = \operatorname{Ker}(\alpha) \subset E$ be the corresponding hyperplane in E. Consider the disconnected space

$$E^0 = E \setminus \bigcup_{\alpha \in R} H_\alpha \qquad (\text{Note: } H_\alpha = H_{-\alpha})$$

The connected components of E^0 are called <u>chambers</u>. Let us choose a chamber, say $\mathcal{C}^0 \subset E^0$, and refer to it as <u>the fundamental chamber.</u> This choice defines the notion of positive and negative roots as follows.

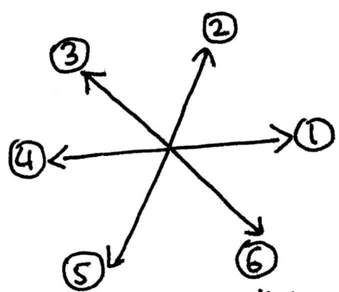
[Note : for any connected component $\mathcal{C} \subset E^\circ$, and $\alpha \in R$, either $\alpha(x) > 0 \ \forall x \in \mathcal{C}$; or $\alpha(x) < 0 \ \forall x \in \mathcal{C}$. Since otherwise, H_α will cross through $\mathcal{C}$.]

$$R_+ := \{\alpha \in R \mid \alpha(x) > 0 \ \forall x \in \mathcal{C}^\circ\},$$

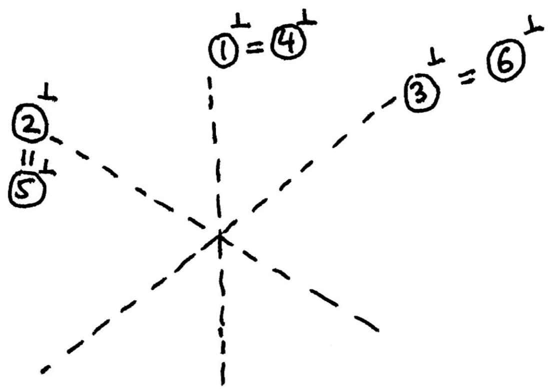
$$R_- := \{\alpha \in R \mid \alpha(x) < 0 \ \forall x \in \mathcal{C}^\circ\}.$$ Then we have

$$R = R_+ \sqcup R_- \quad \text{and} \quad R_- = -R_+.$$

Example (A_2)

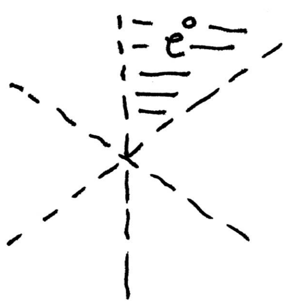


(roots in E^*)



hyperplanes in E. E° has 6 connected components

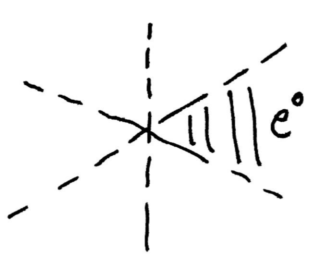
If we choose $\mathcal{C}^\circ$ to be



then ①, ② and ③ $= R_+$

④, ⑤ and ⑥ $= R_-$

If we choose $\mathcal{C}^\circ$ to be



then ①, ② and ⑥ $= R_+$

③, ④ and ⑤ $= R_-$.

2. Simple roots.

Given $\mathcal{C} \subset E^{\circ}$ (a connected component) and $\alpha \in R$, we say <u>α is a wall of $\mathcal{C}$</u> if

$$\begin{cases} \bullet \ \alpha(x) > 0 \quad \forall x \in \mathcal{C} \quad \text{and} \\ \bullet \ \overline{\mathcal{C}} \cap H_\alpha \text{ is of dimension} = \dim E - 1. \end{cases}$$

$\overline{\mathcal{C}}$: closure in standard topology of $\mathbb{R}^n$

Having chosen a fundamental chamber $\mathcal{C}^{\circ}$, let $\Delta \subset R_+$ be the set of walls of $\mathcal{C}^{\circ}$. To preserve notation we will write

$$\{\alpha_i\}_{i \in I} = \Delta. \quad (\text{called } \underline{\text{simple roots}}).$$

(I: indexing set)

Lemma. Let $i, j \in I$ and assume $i \neq j$. Let $c \in \mathbb{R}_{>0}$.

Then $\alpha_i - c\alpha_j$ is not a root.

Proof. As α_i and α_j are not proportional; if $\alpha = \alpha_i - c\alpha_j \in R$,

then either $\alpha \geq 0$ on $\overline{\mathcal{C}^{\circ}}$

or $\alpha \leq 0$ on $\overline{\mathcal{C}^{\circ}}$. However, α is < 0 on $H_{\alpha_i} \cap \overline{\mathcal{C}^{\circ}}$

and $\alpha > 0$ on $H_{\alpha_j} \cap \overline{\mathcal{C}^{\circ}}$. This is a contradiction. $\square$

Corollary. - For $i \neq j$; $i, j \in I$, let $a_{ij} := \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)} = \alpha_j(\alpha_i^\vee)$.

Then $a_{ij} \in \mathbb{Z}_{\leq 0}$.

Proof. $a_{ij} \in \mathbb{Z}$ by axiom of integrality (3) of root systems.

Now $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i \in R$ and hence, by the lemma above, $a_{ij} \leq 0$.

3. Proposition. - $\{\alpha_i\}_{i \in I}$ forms a basis of E^*.

Proof. - Let us first prove that this set is linearly independent.

Assume a linear relation existed $\sum_{t=1}^{p} c_t \alpha_{i_t} = 0$ where all c_t's are non-negative, and, say $c_1 = 1$. Then $-\alpha_{i_1} = \sum_{t=2}^{p} c_t \alpha_{i_t}$. The left-hand side of this equation is < 0 on $\mathcal{C}^\circ$ while the right-hand side is ≥ 0 - contradiction.

Thus if a linear relation existed, among $\{\alpha_i\}_{i \in I}$, it must be of the form $\beta = \sum_{j=1}^{p} c_j \alpha_{i_j} = \sum_{k=p+1}^{q} d_k \alpha_{i_k}$; $c_1, \dots, c_p, d_{p+1}, \dots, d_q \in \mathbb{R}_{>0}$

and $\alpha_{i_1}, \dots, \alpha_{i_p}$ are distinct from $\alpha_{i_{p+1}}, \dots, \alpha_{i_q}$.

But then $(\beta, \beta) = \sum_{j,k} c_j d_k (\alpha_{i_j}, \alpha_{i_k}) \leq 0$, contradicting positive definiteness, unless $\beta = 0$ - in which case we obtain a linear relation with $\mathbb{R}_{>0}$ coefficients - the case we already excluded.

Now we claim that $\{\alpha_i\}_{i\in I}$ spans E^*. As R spans E^*, it is enough to show that $\forall\, \alpha \in R$, $\{\alpha\} \cup \{\alpha_i\}_{i\in I}$ is linearly dependent. Assume, on the contrary, that there exists $\alpha \in R$ such that $\{\alpha\} \cup \{\alpha_i\}_{i \in I}$ is linearly independent. Then we can find $x, y \in E$ s.t. $\alpha_i(x) > 0$; $\alpha_i(y) > 0$ $\forall\, i \in I$

$\alpha(x) > 0$ and $\alpha(y) < 0$

$\Rightarrow$ H_α passes through $\mathcal{C}^\circ$, a contradiction. □

4. Cartan matrix and Dynkin diagram. -

Let $A = (a_{ij})_{i,j\in I}$ be the integer matrix where

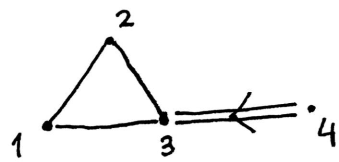
$$a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)} = \alpha_j(\alpha_i^\vee) = \begin{cases} 2 & \text{if } i=j \\ \in \mathbb{Z}_{\leq 0} & \text{if } i \neq j \end{cases}$$

By our rank 2 classification, $\forall\, i \neq j$, we have the following possibilities for (a_{ij}, a_{ji}) (up to switching the role of i & j).

$$(a_{ij}, a_{ji}) = \begin{cases} (0,0) & - \text{ draw as } \underset{i}{\bullet} \quad \underset{j}{\bullet} \quad \text{not connected.} \\ (-1,-1) & - \quad '' \quad '' \quad \underset{i}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\underset{j}{\bullet} \quad \text{'simple edge'} \\ (-2,-1) & - \quad '' \quad '' \quad \underset{i}{\bullet}\Longleftarrow\underset{j}{\bullet} \quad \text{'double edge and } |\alpha_j|^2 > |\alpha_i|^2\text{'} \\ (-3,-1) & - \quad '' \quad '' \quad \underset{i}{\bullet}\Lleftarrow\underset{j}{\bullet} \quad \text{'triple edge and } |\alpha_j|^2 > |\alpha_i|^2\text{'} \end{cases}$$

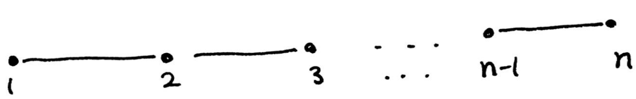
The Dynkin diagram associated to the root system R is then a graph on the vertex set I and edges drawn according to the rule given above.

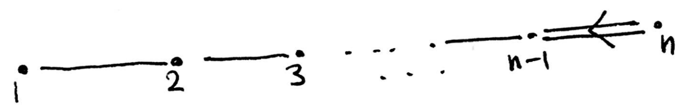
e.g. $A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix} \longleftrightarrow$ Dynkin diagram

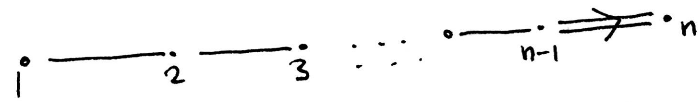


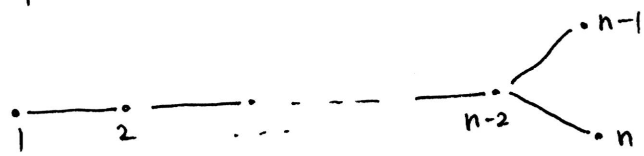
(Note - This "Cartan matrix" does not arise from any root system, since it is not positive-definite $A \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.)

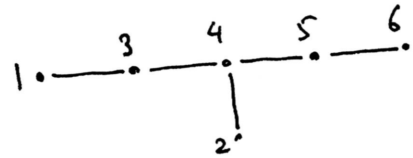
5. Root systems are completely classified by their Dynkin diagrams:

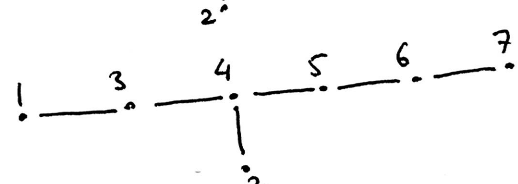
A_n 

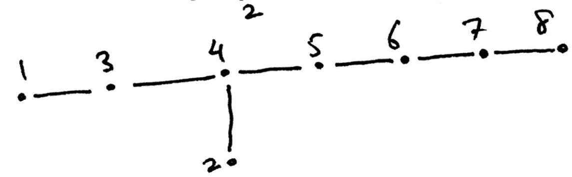
B_n 

C_n 

D_n 

E_6 

E_7 

E_8 

F_4 

G_2