Lecture 7

0. Recall: E = f.d. real vector space. $(\cdot,\cdot) : E \times E \to \mathbb{R}$ positive-def. form

$\nu : E^* \xrightarrow{\sim} E$; $(\cdot,\cdot) : E^* \times E^* \to \mathbb{R}$; $\forall \alpha \in E^* \setminus \{0\}$

$\left(\begin{array}{l} (\nu(\alpha), \phi) = \alpha(\phi) \\ \forall \alpha \in E^*, \phi \in E \end{array} \right)$ $\quad \left((\nu(\alpha), \nu(\beta)) =: (\alpha, \beta) \right)$ $\quad \alpha^\vee := \frac{2}{(\alpha,\alpha)} \cdot \nu(\alpha) \in E$

- $R \subset E^* \setminus \{0\}$ (finite) root system, i.e.
  (1) R spans E^* ; (2) $\alpha, c\alpha \in R \Rightarrow c = \pm 1$
  (3) $\alpha, \beta \in R \Rightarrow \beta(\alpha^\vee) \in \mathbb{Z}$; (4) $\alpha \in R \Rightarrow s_\alpha(R) \subset R$.

  Here $s_\alpha : E^* \to E^*$.
  $\gamma \mapsto \gamma - \gamma(\alpha^\vee) \cdot \alpha$

- Choose a connected component $\mathcal{C}^\circ \subset E^\circ = E \setminus \bigcup_{\alpha \in R} H_\alpha$ where

  $H_\alpha = \mathrm{Ker}(\alpha) \subset E$. (or $\alpha^\perp$) $\quad R_\pm := \{\alpha \in R \mid \pm\alpha(x) > 0 \ \forall x \in \mathcal{C}^\circ\}$ (positive/negative roots.)

  $\{\alpha_i\}_{i \in I} \subset R_+$, called simple roots, are the ones defining walls of $\mathcal{C}^\circ$.

- $A = (a_{ij})_{i,j \in I}$ Cartan matrix, $a_{ij} := \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$
  $= \alpha_j(\alpha_i^\vee) \left(\begin{array}{l} \in \mathbb{Z}_{\leq 0} \\ \text{if } i \neq j \end{array} \right)$

Dynkin diagram on Vertex set I:

(a_{ij}, a_{ji})	$(0,0)$	$(-1,-1)$	$(-2,-1)$	$(-3,-1)$
Edge b/w i,j	$i \quad j$ no edge	i — j	$i \Leftarrow j$	$i \Lleftarrow j$

Last time we proved that $\{\alpha_i\}_{i\in I}$ form a basis of E^*.

Thus, $R_+ = \left(\sum_{i\in I} \mathbb{Z}_{\geq 0}\, \alpha_i\right) \cap R$ (and $R_- = -R_+$ as before).

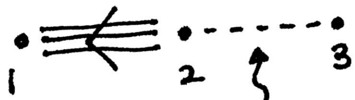
As a consequence of this, we conclude that $A = (a_{ij})_{i,j\in I}$ must be positive-definite. More precisely, if $d_i = \frac{(\alpha_i, \alpha_i)}{2} \in \mathbb{R}_{>0}$ $(\forall\, i \in I)$

then $[d_i a_{ij}]_{i,j\in I}$ is the matrix of $(\cdot,\cdot)$ on E^*, written in the basis $\{\alpha_i\}_{i\in I}$.

1. Cor. If $\underline{u} = (u_i)_{i\in I} \in \mathbb{R}^I$ is such that $\underline{u} \geq 0$ (meaning, each component is ≥ 0) and $A\underline{u} \leq 0$ then $\underline{u} = 0$.

2. Now we assume that the Dynkin diagram Γ of the root system R is connected. We want to prove that it must be one of diagrams listed on §5 of Lecture 6 (page 7). The argument follows the exclusion principle - for which we use the Corollary 1 above.

- If Γ contains a triple edge then $\Gamma = \bullet \Lleftarrow \bullet$ (type G_2).

  Proof. By connectedness, if Γ contains a triple edge $(1 \Lleftarrow 2)$ and another vertex, then Γ must contain either 

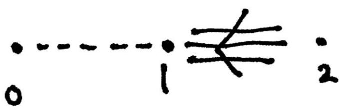
edge of some type

$\rightsquigarrow \begin{bmatrix} 2 & -3 & -c \\ -1 & 2 & -a \\ -d & -b & 2 \end{bmatrix}$ (submatrix of Γ) $(ab \neq 0)$

(1 & 3 may be linked as well)

Take $\underline{u} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, then $A\underline{u} = \begin{bmatrix} -c \\ 1-a \\ 2-2b-3d \end{bmatrix} \leq 0$

contradicts the corollary.

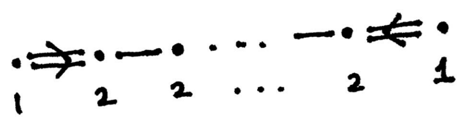
or 

$\rightsquigarrow \begin{bmatrix} 2 & -a & -c \\ -b & 2 & -3 \\ -d & -1 & 2 \end{bmatrix}$. Take $\underline{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

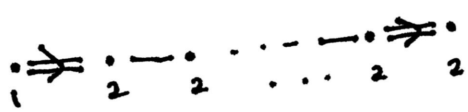
$(ab \neq 0)$

Then $A\underline{u} = \begin{bmatrix} 2-2a-c \\ 1-b \\ -d \end{bmatrix} \leq 0$.

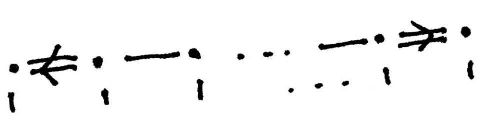
Notation: below, I will just write entries of $\underline{u}$ that contradict the Corollary, on a diagram, to conclude that it cannot be a sub-diagram of Γ.

- 

$= \underline{u}$

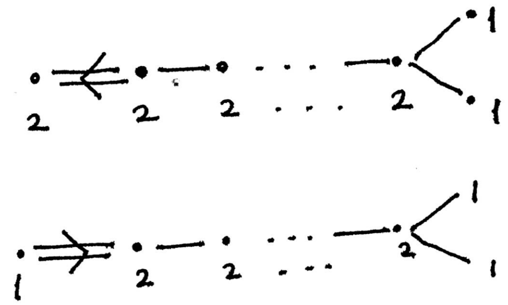


$= \underline{u}$

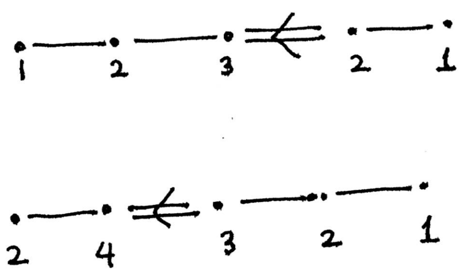


$= \underline{u}$

$\Rightarrow$ Γ can have at most one double edge.

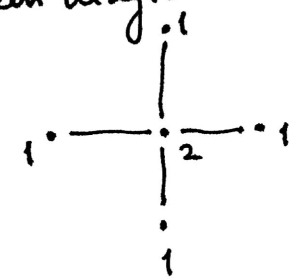
- 

$\Rightarrow$ if Γ has a double edge it cannot have a vertex of degree ≥ 3.

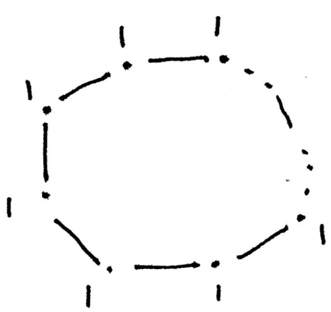
- 

$\Rightarrow$ If Γ has a double edge it can only be B_n, C_n or F_4.

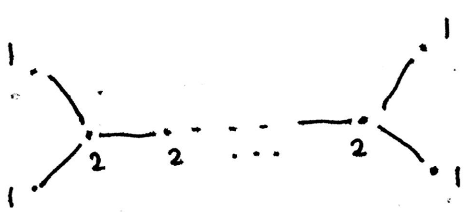
3. Now we are left with those Dynkin diagrams which are entirely composed of simple edges. Again



$\Rightarrow$ Γ cannot have a vertex of degree 4 or higher.



$\Rightarrow$ Γ cannot have a cycle



$\Rightarrow$ Γ can have at most one vertex of degree 3.

Thus Γ can only be of the following form :

$$\begin{tikzcd}[column sep=small]
i_1 \arrow[r, dash] & i_2 \arrow[r, dash] & \cdots \arrow[r, dash] & i_p \arrow[r, dash] & u \arrow[r, dash] \arrow[d, dash] & j_q \arrow[r, dash] & j_{q-1} \cdots \arrow[r, dash] & j_1 \\
 & & & & k_r \arrow[d, dash] & & & \\
 & & & & \vdots \arrow[d, dash] & & & \\
 & & & & k_1 & & &
\end{tikzcd}$$

$p, q, r \geq 0$.

Assume without loss of generality that $p \geq q \geq r$.

Set $x = \sum_{t=1}^{p} t \cdot \alpha_{i_t}$; $y = \sum_{t=1}^{q} t \cdot \alpha_{j_t}$; $z = \sum_{t=1}^{r} t \cdot \alpha_{k_t}$;

$w = \alpha_u$.

Then (i) x, y and z are mutually orthogonal.

(ii) $|x|^2 = p(p+1)$; $|y|^2 = q(q+1)$; $|z|^2 = r(r+1)$

$(x, w) = -p$; $(y, w) = -q$; $(z, w) = -r$

$|w|^2 = 2$.

As $w \notin \mathrm{Span}\{x, y, z\}$; $\left(\begin{array}{l}\text{distance between } w \text{ and } \mathbb{R}\text{-span} \\ \text{of } x, y, z\end{array}\right)^2 > 0$.

i.e. $$|w|^2 - \frac{(x,w)^2}{|x|^2} - \frac{(y,w)^2}{|y|^2} - \frac{(z,w)^2}{|z|^2} > 0$$

$$\Rightarrow \boxed{\frac{1}{p+1} + \frac{1}{q+1} + \frac{1}{r+1} > 1} \quad - (3.1)$$

As $p \geq q \geq r$; L.H.S. of (3.1) $\leq \dfrac{3}{r+1}$. So we get

$$1 < \frac{3}{r+1} \quad \text{i.e.} \quad r < 2 \quad \Rightarrow \quad r = 0 \text{ or } 1.$$

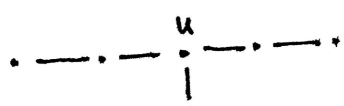
If $r = 0$, we get no constraint for p, q and Γ is of $\boxed{\text{type } A}$ (simple chain.)

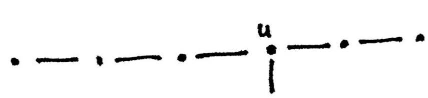
If $r = 1$, $\dfrac{1}{p+1} + \dfrac{1}{q+1} > \dfrac{1}{2}$. Using the same logic as $(p \geq q \geq 1)$

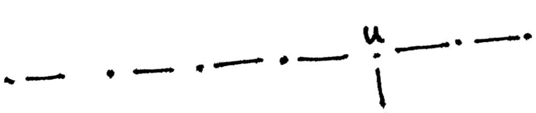
before $\dfrac{2}{q+1} \geq \dfrac{1}{p+1} + \dfrac{1}{q+1} > \dfrac{1}{2} \Rightarrow q+1 < 4$, i.e.

$q = 1$ or 2. $q = 1$ gives $\begin{array}{c} i_1 - \cdots - i_p - u - j_1 \\ \qquad\qquad\quad | \\ \qquad\qquad\quad k_1 \end{array}$ $\boxed{\text{type } D}$

while $q = 2 \Rightarrow \dfrac{1}{p+1} > \dfrac{1}{2} - \dfrac{1}{3} = \dfrac{1}{6}$. Hence $p = 2, 3, 4$.
$(p \geq 2)$

$p = 2,\ q = 2,\ r = 1$  E_6

$p = 3,\ q = 2,\ r = 1$  E_7

$p = 4,\ q = 2,\ r = 1$  E_8 .