Lecture 8 - Weyl group.

0. Recall the notations for root system $R \subset E^* \setminus \{0\}$. (see, e.g, Lecture 7, §0, page 1).

The <u>Weyl group of R</u>, denoted by W, is the subgroup of $GL(E^*)$ (or $GL(E)$) generated by $\{s_\alpha\}_{\alpha \in R}$.

Remarks. – (i) As R spans E^*, and W preserves R (by axiom (4) of root systems $s_\alpha(R) \subset R \ \forall \alpha \in R$.), we conclude that W can be viewed as a subgroup of the permutation group on R.

Hence $|W| < \infty$.

(ii) W preserves $(\cdot,\cdot)$ on E (or E^*).

Proof. Let $\alpha \in R$; $\phi, \psi \in E$. Then

$$(s_\alpha(\phi), s_\alpha(\psi)) = (\phi - \alpha(\phi)\alpha^\vee, \psi - \alpha(\psi)\alpha^\vee)$$
$$= (\phi,\psi) + \alpha(\phi)\alpha(\psi)(\alpha^\vee,\alpha^\vee)$$
$$- \alpha(\phi)(\psi,\alpha^\vee) - \alpha(\psi)(\phi,\alpha^\vee)$$

[$(\psi,\alpha^\vee) = \frac{2}{(\alpha,\alpha)}\alpha(\psi)$ and $(\phi,\alpha^\vee) = \frac{2}{(\alpha,\alpha)}\alpha(\phi)$ by definition of $\alpha^\vee$. Also, $(\alpha^\vee,\alpha^\vee) = \frac{4}{(\alpha,\alpha)}$.]

$$\Rightarrow (s_\alpha(\phi), s_\alpha(\psi)) = (\phi,\psi) + \alpha(\phi)\alpha(\psi)\left(\frac{4}{(\alpha,\alpha)} - \frac{2}{(\alpha,\alpha)} - \frac{2}{(\alpha,\alpha)}\right)$$
$$= (\phi,\psi).$$

□

1. Therefore, $W \circlearrowright E$ preserves the hyperplanes $\{H_\alpha\}_{\alpha\in R}$ $\left(w(H_\alpha) = H_{w(\alpha)}\right)$ and hence $W \circlearrowright$ set of connected components of $E^\circ = E \setminus \bigcup_{\alpha\in R} H_\alpha$.

Let $\mathcal{C}^\circ \subset E^\circ$ be the fundamental chamber and $\{\alpha_i\}_{i\in I}$ corresponding simple roots. We will first prove that W is generated by $\{s_i = s_{\alpha_i}\}_{i\in I}$. For this, we need the following lemma. (For now, let $W' \leq W$ be the subgp. generated by $\{s_i\}_{i\in I}$)

Lemma. – Let $y \in E$. Then $\exists\, w \in W'$ such that
$$w(y) \in \overline{\mathcal{C}^\circ}.$$

Proof. – Choose $a \in \mathcal{C}^\circ$ and let $W'\cdot y$ be the orbit of y under W'. Since this is a finite set, we can choose
$$y_0 \in W'\cdot y \quad \text{such that}$$
$$\text{distance}(a, y_0) \leq \text{distance}(a, y') \quad \forall\, y' \in W'\cdot y$$

Claim: $\forall\, i \in I$, $\alpha_i(y_0) \geq 0$. (i.e., $y_0 \in \overline{\mathcal{C}^\circ}$).

Pf. of the claim. – Since $\text{distance}(a, y_0)^2 \leq \text{distance}(a, s_i(y_0))^2$, we get:
$$|a - y_0|^2 \leq |a - s_i(y_0)|^2$$
$$\equiv \quad \cancel{|a|^2} + \cancel{|y_0|^2} - 2(a, y_0) \leq \cancel{|a|^2} + \cancel{|s_i(y_0)|^2} - 2(a, s_i(y_0))$$

(W preserves lengths)

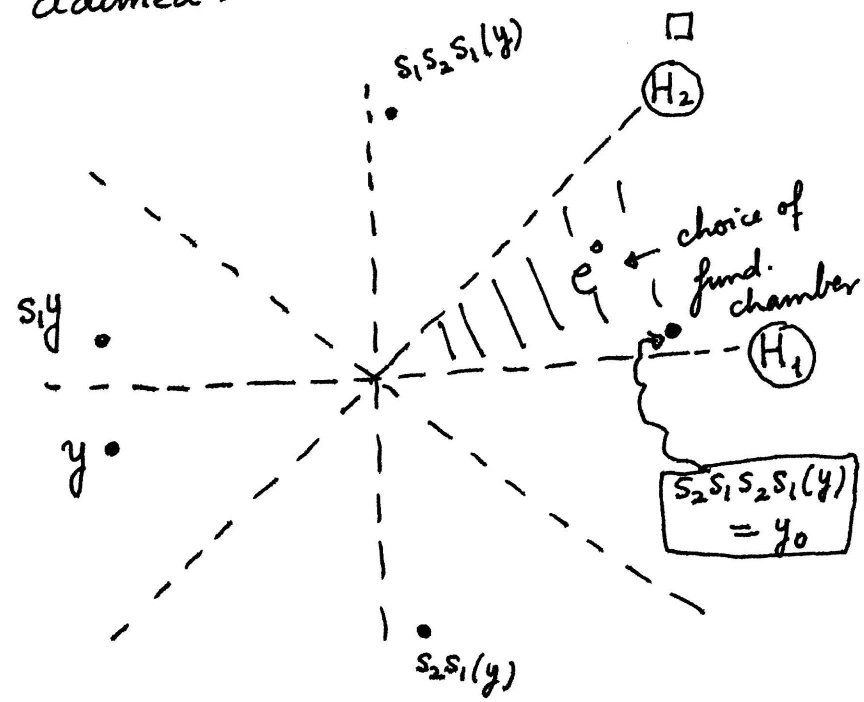
i.e. $\quad (a, y_0 - s_i(y_0)) \geq 0 \quad$ (recall $s_i(y_0) = y_0 - \alpha_i(y_0)\alpha_i^\vee$)

$$\alpha_i(y_0)\,(a, \alpha_i^\vee) \geq 0$$

As $a \in e^\circ$; $\quad (a, \alpha_i^\vee) = \dfrac{2}{(\alpha_i, \alpha_i)}\,\alpha_i(a) > 0$. Hence we get that $\alpha_i(y_0) \geq 0$ as claimed. $\square$

2. Example.

Lemma 1. proves that any point y can be moved to e° by using only the reflections through the walls of e° (denoted here as H_1 and H_2).



(hyperplane arrangement in type B_2)

3. Corollaries of Lemma 1.

(i) W' action on $\pi_0(E^\circ)$ is transitive. That is, given any chamber $e \subset E^\circ$, we can find $w \in W'$ such that $w(e) = e^\circ$.

(ii) $\quad R = \bigcup_{i \in I} W' \cdot \alpha_i \quad$. By definition, $\bigcup_{i \in I} W' \cdot \alpha_i \subset R$.

Let $\alpha \in R$. Let $\mathcal{C} \subset E^\circ$ be a chamber s.t. α is a wall of $\mathcal{C}$. Let $w \in W'$ be such that $w(\mathcal{C}) = \mathcal{C}^\circ$. As walls of $\mathcal{C}^\circ$ are precisely $\{\alpha_i\}_{i\in I}$ we have : $w(H_\alpha) = H_{\alpha_j}$ for some $j \in I$. Hence $w(\alpha) = \alpha_j$.

(iii) $W' = W$. Since W is generated by $\{s_\alpha\}_{\alpha\in R}$, it is enough to show that $s_\alpha \in W'$ $\forall\, \alpha \in R$. By (ii) above $\alpha = w(\alpha_j)$ for some $w \in W'$ and $j \in I$. But then

$$s_\alpha = s_{w(\alpha_j)} = w\, s_j\, w^{-1} \in W'.$$

4. Thus, $W = \langle s_i \rangle_{i\in I}$. Recall that $s_i^2 = 1$ $\forall\, i \in I$.

Definition. Given $w \in W$, the <u>length of w</u>, denoted by $\ell(w)$, is defined as :

$$\ell(w) := \text{MIN} \{ k \text{ s.t. } \exists\, i_1, \ldots, i_k \in I \text{ with } w = s_{i_1} \cdots s_{i_k} \}$$

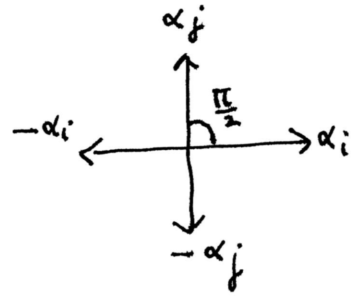
e.g. $\ell(w) = 0 \iff w = e$

$\ell(w) = 1 \iff w = s_i$ for some $i \in I$.

An expression $w = s_{i_1} \cdots s_{i_\ell}$ is called <u>a reduced expression</u> when $\ell = \ell(w)$.

5. Rank 2 relations. – Using our classification of rank 2 root systems, we obtain the following relations.

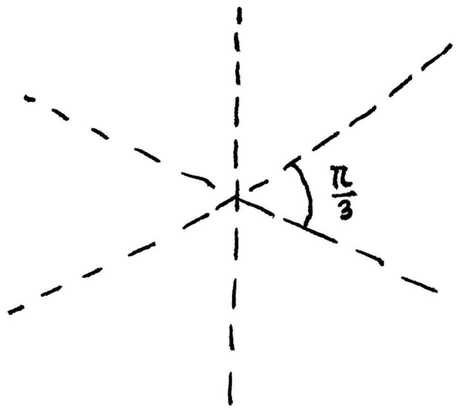
$A_1 \times A_1$:



: $s_i s_j = s_j s_i$.

or $(s_1 s_2)^2 = e$.

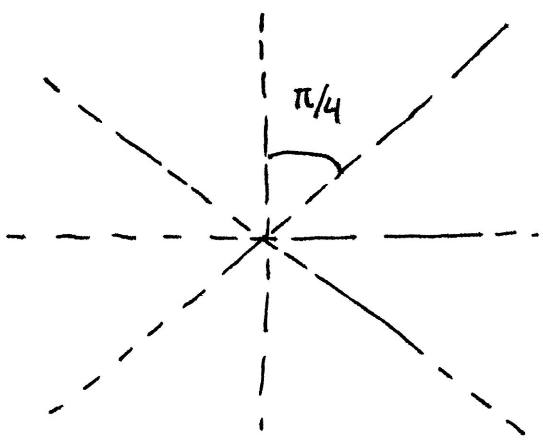
A_2 :



$\Rightarrow$ $s_1 s_2$ = rotation by $\frac{2\pi}{3}$

$\Rightarrow$ $(s_1 s_2)^3 = e$.

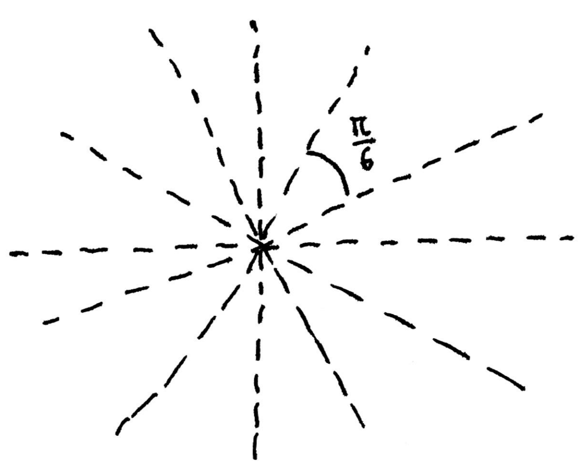
B_2 :



$\Rightarrow$ $s_1 s_2$ = rotation by $\frac{2\pi}{4}$

$\Rightarrow$ $(s_1 s_2)^4 = e$.

G_2 :



$s_1 s_2$ = rotation by $\frac{2\pi}{6}$

$\Rightarrow$

$\Rightarrow$ $(s_1 s_2)^6 = e$.

6. Lemma. – Let $\alpha \in R_+$ and $i \in I$. Then

$$s_i(\alpha) \in R_- \iff \alpha = \alpha_i .$$

Proof. – $s_i(\alpha) = \alpha - \alpha(\alpha_i^\vee)\cdot\alpha_i \in R_-$

$$\alpha = \sum_{j\in I} n_j \alpha_j \in R_+ \quad (n_j \in \mathbb{Z}_{\geq 0}).$$

This is possible $\iff n_j = 0 \ \forall j \neq i$ i.e. $\alpha = \alpha_i$ (only positive root proportional to α_i). $\square$

7. Exchange property.

Proposition. – Let $w \in W$ and $i \in I$. Then the following conditions are equivalent:

(1) $\ell(ws_i) < \ell(w)$

(2) $w(\alpha_i) \in R_-$

(3) For any reduced expression of w : $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$, (i.e. $\ell = \ell(w)$)

$\exists j \in \{1, \ldots, \ell\}$ such that

$$s_{i_j} s_{i_{j+1}} \cdots s_{i_\ell} = s_{i_{j+1}} \cdots s_{i_\ell} \cdot s_i$$

Proof. $\underline{(2) \Rightarrow (3)}$: Assume $w(\alpha_i) \in R_-$ and w is written as a product of simple reflections $w = s_{i_1} \cdots s_{i_\ell}$; where $\ell = \ell(w)$.

Set $\beta_j = s_{i_{j+1}} \cdots s_{i_\ell}(\alpha_i) \quad (0 \leq j \leq \ell)$

$\beta_\ell = \alpha_i \in R_+$ and $\beta_0 = w(\alpha_i) \in R_- \Rightarrow \exists j ;\ 1 \leq j \leq \ell,$

so that $\beta_{j-1} \in R_-$ and $\beta_j \in R_+$.

Since $\beta_{j-1} = s_{i_j}(\beta_j)$ this can only happen (see Lemma 6 above)

when $\beta_j = \alpha_{i_j}$. That means, $\alpha_{i_j} = \beta_j = \underbrace{\boxed{s_{i_{j+1}} \cdots s_{i_\ell}}}_{\text{call it } v \in W}(\alpha_i)$

and hence
$$s_{i_j} = v\, s_i\, v^{-1}$$
$$= s_{i_{j+1}} \cdots s_{i_\ell}\, s_i\, s_{i_\ell} \cdots s_{i_{j+1}}$$

i.e. $s_{i_j} s_{i_{j+1}} \cdots s_{i_\ell} = s_{i_{j+1}} \cdots s_{i_\ell} \cdot s_i$ as we wanted.

<u>(3) ⇒ (1)</u>: Let $\ell = \ell(w)$ and $w = s_{i_1} \cdots s_{i_\ell}$ a reduced expression for w. By (3), $w = s_{i_1} \cdots s_{i_{j-1}} s_{i_{j+1}} \cdots s_{i_\ell} \cdot s_i$ (for some j)

$\Rightarrow \; w s_i = s_{i_1} \cdots s_{i_{j-1}} s_{i_{j+1}} \cdots s_{i_\ell} \;\Rightarrow\; \ell(ws_i) \leq \ell - 1 < \ell(w)$.

<u>(1) ⇒ (2)</u>: Assume the contrary: $w(\alpha_i) \in R_+$. Therefore, $ws_i(\alpha_i) \in R_-$. We have already proved that (2) ⇒ (3) ⇒ (1) and we apply it to $u = ws_i$ to conclude that $\ell(us_i) < \ell(u)$

i.e. $\ell(w) < \ell(ws_i)$.

□