Lecture 9

0. Recall the notations and definitions regarding root systems and associated Weyl group. (Lectures 7 & 8, §0.)

Last time we proved the following properties of the Weyl group:

(i) W acts transitively on the set of chambers.

(ii) W is generated by $\{s_i\}_{i\in I}$ - (reflections through the walls of the fundamental chamber $\mathcal{C}^\circ$).- called <u>simple reflections</u>.

$\rightsquigarrow$ length function: $\forall\, w\in W$, $\ell(w)\in \mathbb{Z}_{\geq 0}$ defined as:

$$\ell(w) := \text{MIN}\left\{ k \;\middle|\; \begin{array}{c}\exists\, i_1,\dots,i_k\in I \text{ with} \\ w = s_{i_1}\cdots s_{i_k}\end{array}\right\}$$

An expression of w in terms of simple reflections $w = s_{i_1}\cdots s_{i_\ell}$ is called <u>reduced</u> if $\ell = \ell(w)$.

(iii) For $w\in W$ and $i\in I$, TFAE:

- $\ell(ws_i) < \ell(w)$
- $w(\alpha_i)\in R_-$
- for any reduced expression $w = s_{i_1}\cdots s_{i_\ell}$, there exists j, $1\leq j\leq \ell$, s.t.

$$s_{i_j}s_{i_{j+1}}\cdots s_{i_\ell} = s_{i_{j+1}}\cdots s_{i_\ell}s_i \qquad \left[\begin{array}{c}\text{Exchange}\\ \text{property}\end{array}\right]$$

1. Consequences of the exchange property (Prop. 7 of Lecture 8).

~~(a)~~ W acts freely on the set of chambers (i.e. $\pi_0(E^\circ) =$ connected components of $E^\circ = E \setminus \bigcup_{\alpha \in R} H_\alpha$.)

Let $a \in \mathcal{C}^\circ$, $w \in W$ and assume that $w(a) = a' \in \mathcal{C}^\circ$.
We will show that $\ell(w) = 0$, i.e. $w = e$. If not, choose a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ $(\ell = \ell(w) \geq 1)$. Then $\ell(w s_{i_\ell}) < \ell(w)$, hence $w(\alpha_{i_\ell}) \in R_-$. We conclude

$$\alpha_{i_\ell}(a) > 0 \quad (\text{since } a \in \mathcal{C}^\circ)$$

$$w(\alpha_{i_\ell})(wa) = \alpha_{i_\ell}(a) > 0.$$

$$[W \text{ preserves } E^* \times E \to \mathbb{R}]$$

But $wa = a' \in \mathcal{C}^\circ$ by our hypothesis and $w(\alpha_{i_\ell}) \in R_-$, which is a contradiction.

Remark.- The same argument shows that if $a \in E$, and

$$W_a := \{ w \in W \mid w(a) = a \} \quad (\text{stabilizer}),$$

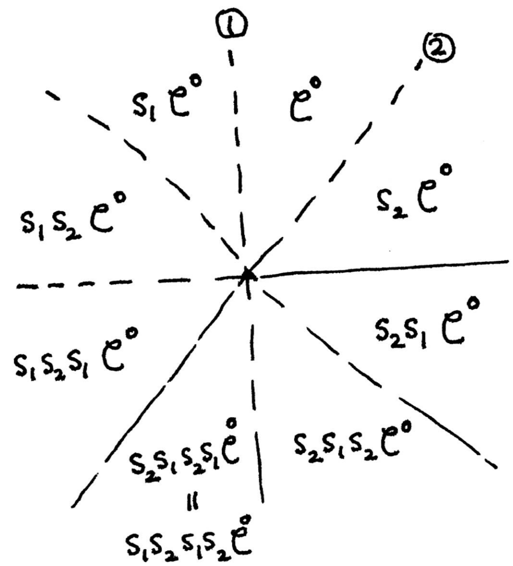
then $\underset{(\text{reduced})}{w = s_{i_1} \cdots s_{i_\ell}} \in W_a \iff s_{i_1}, \ldots, s_{i_\ell} \in W_a$.
(Exercise).

Thus we obtain a bijection

$$\begin{array}{ccc} W & \longleftrightarrow & \pi_0(E^\circ) \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ w & \longrightarrow & w(C^\circ) \end{array}$$

e.g.

(arrangement of type B_2)



$W = \langle s_1, s_2 \rangle$

$= \{ e, s_1, s_2, s_1 s_2, s_2 s_1,$

$s_1 s_2 s_1, s_2 s_1 s_2, s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 \}$

2. A presentation of W. Recall (Lecture 8, §5, page 5) that we obtained the following relations from our rank 2 study

$i \neq j;\ i, j \in I \quad \Rightarrow \quad (s_i s_j)^{m_{ij}} = e$ where m_{ij}'s are given

by:

$a_{ij} a_{ji}$	0	1	2	3
m_{ij}	2	3	4	6

Lemma. Let M be a monoid with 1, and assume we have $T_i \in M$ ($\forall i \in I$) s.t. $\underbrace{T_i T_j T_i \cdots}_{m_{ij}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}}$ $\forall i \neq j$

For any $w \in W$, let $w = s_{i_1} \dots s_{i_\ell}$ be a reduced expression. Then $T_w := T_{i_1} \dots T_{i_\ell} \in M$ is independent of the choice of the reduced expression.

Proof. - Consider the set of all reduced expressiom of w:

$$\mathcal{R}(w) = \{ (i_1, \dots, i_\ell) \in I^\ell \mid w = s_{i_1} \dots s_{i_\ell} \} \quad (\ell = \ell(w)).$$

We will prove the lemma by induction on $\ell(w)$. The cases $\ell(w) = 0$ (and also $\ell(w) = 1$) being obvious.

Let $(i_1, \dots, i_\ell)$ and $(j_1, \dots, j_\ell)$ be two reduced expressions of w. Observation. If $i_1 = j_1$ or $i_\ell = j_\ell$, the identity

$$T_{i_1} \dots T_{i_\ell} = T_{j_1} \dots T_{j_\ell}$$ follows by induction.

Assume this is not the case. As $\ell(w \cdot s_{i_\ell}) < \ell(w)$, using the exchange property on $w = s_{j_1} \dots s_{j_\ell}$; we find another reduced expression

$s_{j_1} \dots \hat{s}_{j_t} \dots s_{j_\ell} s_{i_\ell}$ which has the same last term as $(i_1 \dots i_\ell)$ and (unless $t = 1$) has the same first term as $(j_1, \dots, j_\ell)$. The observation above finishes the proof, unless exchange property changes

$$w = s_{j_1} \dots s_{j_\ell} \rightsquigarrow s_{j_2} \dots s_{j_\ell} s_{i_\ell} = w \quad (\text{i.e. } t = 1 \text{ here}).$$

Similarly (switching the roles of $(i_1 \dots i_\ell) \leftrightarrow (j_1 \cdots j_\ell)$)

$$\begin{array}{c} w = s_{i_1} \cdots s_{i_\ell} \\ \ell(w s_{j_\ell}) < \ell(w) \end{array} \xRightarrow[\text{(exchange)}]{} s_{i_1} \cdots \hat{s}_{i_r} \cdots s_{i_\ell}$$

$\nearrow$ $r \neq 1$ we are done by observation

$\searrow$ $r = 1$

$w = s_{i_2} \cdots s_{i_\ell} s_{j_\ell}$

Continuing this way, either we will be done by the observation above; or the repeated application of the exchange property always skips the first term:

$$\begin{array}{lllll} w = s_{i_1} \cdots s_{i_\ell} & \rightsquigarrow & s_{i_2} \cdots s_{i_\ell} s_{j_\ell} & \cdots \rightsquigarrow & \boxed{\begin{array}{c} \cdots s_{j_\ell} s_{i_\ell} s_{j_\ell} \\ \\ \cdots s_{i_\ell} s_{j_\ell} s_{i_\ell} \end{array}} \\ w = s_{j_1} \cdots s_{j_\ell} & \rightsquigarrow & s_{j_2} \cdots s_{j_\ell} s_{i_\ell} & \cdots \rightsquigarrow & \end{array}$$

m_{i_ℓ, j_ℓ} terms

and we are done by the assumption

$$T_i T_j T_i \cdots = T_j T_i T_j \cdots \quad (m_{ij} \text{ terms})$$

$\square$

3. Prop. W admits the following presentation:

Generators: $s_i \quad (i \in I)$

Relations: $s_i^2 = e \quad \forall i \in I.$

$$(s_i s_j)^{m_{ij}} = e \quad \forall\ i \neq j \in I.$$

Proof. Let G be any group, $f : I \longrightarrow G$ be a set map
$$\begin{array}{ccc} I & \longrightarrow & G \\ \Psi & & \Psi \\ i & \longmapsto & f_i \end{array}$$

s.t. $f_i^2 = e \ (\forall\, i \in I)$ and $(f_i f_j)^{m_{ij}} = e \ \ \forall\, i \neq j \in I$.

We want to prove that f extends to a unique group hom. $g : W \to G$ such that $g(s_i) = f_i$. Uniqueness is clear since W is generated by $\{s_i\}_{i \in I}$. Define $g(w) \in G$ by choosing a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ and $g(w) := f_{i_1} \cdots f_{i_\ell}$ (well-defined by Lemma 2 above).

To prove: g is a group homomorphism. In fact it suffices to show that $g(s_i w) = g(s_i)\, g(w) \quad \forall\, i \in I,\ w \in W$.

Case 1. $\ell(s_i w) > \ell(w)$. In this case, for any reduced expression

$w = s_{i_1} \cdots s_{i_\ell}$; $\quad s_i s_{i_1} \cdots s_{i_\ell}$ is a reduced exp. of $s_i w$.

By defn. $g(s_i w) = f_i f_{i_1} \cdots f_{i_\ell} = g(s_i)\, g(w)$.

Case 2. $\ell(s_i w) < \ell(w)$. Let $u = s_i w$, so that $\ell(s_i u) > \ell(u)$, and hence by Case 1, $g(s_i u) = g(s_i)\, g(u)$
$$\Rightarrow \quad g(u) = (f_i)^{-1} \cdot g(s_i u)$$
$$= f_i \ g(s_i u) \qquad (\text{as } f_i^2 = e).$$

i.e. $g(s_i w) = g(s_i)\, g(w)$ in this case as well. $\square$

4. Definition. – The <u>braid group</u> B_W (associated to W) is defined as the group with the following presentation

Generators : $T_i \quad (i \in I)$

Relations : $\underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ terms}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ terms}}$

5. Example in type A.

Let $\mathbb{R}^{n+1}$ be $(n+1)$-dimensional Euclidean space with its standard inner product $(\underline{x}, \underline{y}) = \sum_{i=1}^{n+1} x_i y_i$.

$E \subset \mathbb{R}^{n+1}$ be the kernel of the linear form $\underline{x} \mapsto \sum_{i=1}^{n+1} x_i$.

$E^* \ni \alpha_{ij} = x_i - x_j$ (linear form on $\mathbb{R}^{n+1}$ restricted to E)

$(i \neq j \; ; \; 1 \leq i, j \leq n+1)$

$$R = \{\alpha_{ij}\}_{\substack{i \neq j \\ 1 \leq i,j \leq n+1}} \subset E^* .$$

$$H_{ij} = \mathrm{Ker}(\alpha_{ij}) = \left\{ \underline{x} \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} \sum_{k=1}^{n+1} x_k = 0 \\ x_i = x_j \end{array} \right\}$$

$$E^0 = E \setminus \bigcup_{1 \leq i < j \leq n+1} H_{ij} = \left\{ \underline{x} \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} \sum_{k=1}^{n+1} x_k = 0 \\ x_i\text{'s are all distinct} \end{array} \right\}$$

$$\mathcal{C}^\circ = \left\{ \underline{x} \text{ as before s.t. } x_1 > x_2 > \dots > x_{n+1} \right\}$$

(Fundamental Chamber)
Chosen to be

$$\Rightarrow \quad R_+ = \left\{ \alpha_{ij} \;\middle|\; 1 \le i < j \le n+1 \right\} \; ; \quad R_- = -R_+ .$$

Simple roots $\left\{ \alpha_{i,i+1} \;\middle|\; 1 \le i \le n \right\}$. Cartan Matrix $= \begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}$

Easy check: $\quad s_{\alpha_{ij}} =$ flip of i^{th} & j^{th} coordinate.

$$s_i = s_{\alpha_{i,i+1}} = (i \;\; i+1) .$$

Prop 3 in this context gives a presentation of S_{n+1}

$$= \Big\langle \underbrace{s_i \quad 1 \le i \le n}_{\text{Generators}} \;\Big|\; \underbrace{\begin{array}{ll} s_i^2 = e & \forall i \\ s_i s_j = s_j s_i & |i-j| > 1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \end{array}}_{\text{relations}} \Big\rangle$$

Associated braid group = Artin's braid group on $n+1$ strands

$$= \Big\langle \underbrace{T_i \quad 1 \le i \le n}_{\text{Generators}} \;\Big|\; \begin{array}{ll} T_i T_j = T_j T_i & \text{if } |i-j| > 1 \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & \end{array} \Big\rangle$$