DYNAMICAL YANG–BAXTER EQUATIONS

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Abstract. These are notes from the seminar on dynamical Weyl group, organized during Spring 2011 by the author and V. Toledano Laredo.

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1. Intertwiners and fusion operator for simple Lie algebra $\mathfrak{g}$

We introduce the intertwiners, fusion operator and exchange operator in this section. The main reference for this part is [3, Chapter 3,5]. [1]

1.1. **Notations.** Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and $R \subset \mathfrak{h}^*$ be the set of roots associated with the pair $(\mathfrak{g}, \mathfrak{h})$. Let $\Delta = \{\alpha_i : i \in I\}$ be a base of R and let $R_\pm$ denote the set of positive/negative roots.

We have the triangular decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

Let $\mathfrak{b}_\pm := \mathfrak{h} \oplus \mathfrak{n}_\pm$ be the Borel subalgebras.

Let $\langle ., . \rangle$ be an invariant, non–degenerate, symmetric form on $\mathfrak{g}$ which induces an isomorphism $\nu : \mathfrak{h}^* \to \mathfrak{h}$. Define $d_i := \frac{(\alpha_i, \alpha_i)}{2}$ and $h_i := d_i^{-1}\nu(\alpha_i)$. $\Delta^\vee := \{h_i : i \in I\} \subset \mathfrak{h}$ is the set of simple coroots. Let W be the Weyl group generated by simple reflections $s_i (i \in I)$. We will denote by $\theta \in R_+$ the longest root.

[1] based on the talk by Nate Bade

1.2. **Verma Modules.** For $\lambda \in \mathfrak{h}^*$ let M_λ denote the Verma module of highest weight λ, defined as:

$$M_\lambda := \mathrm{Ind}_{\mathfrak{b}_+}^{\mathfrak{g}} \mathbb{C}\mathbf{1}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}\mathbf{1}_\lambda$$

where $\mathbb{C}\mathbf{1}_\lambda$ is one dimensional $\mathfrak{b}_+$ module defined by:

$$\mathfrak{n}_+\mathbf{1}_\lambda = 0 \qquad\qquad h\mathbf{1}_\lambda = \lambda(h)\mathbf{1}_\lambda$$

Alternately, M_λ can be defined as a quotient of $U(\mathfrak{g})$ by the left ideal generated by $\{x \in \mathfrak{n}_+, h - \lambda(h) : h \in \mathfrak{h}\}$.

1.3. **Shapovalov form.** There exists a unique bilinear form $(.,.)$ on M_λ satisfying:

(1) $(\mathbf{1}_\lambda, \mathbf{1}_\lambda) = 1$
(2) $(e_i v, w) = (v, f_i w)$
(3) $(h_i v, w) = (v, h_i w)$
(4) $(f_i v, w) = (v, e_i w)$

We record some properties of $(.,.)$ in the following:

Proposition. (1) *For $u \in M_\lambda[\mu]$ and $v \in M_\lambda[\nu]$ such that $\mu \neq \nu$ we have*

$$(u, v) = 0$$

Thus $(.,.)$ decomposes as direct sum of its restriction to the weight spaces of M_λ.

(2) *M_λ is irreducible if, and only if $(.,.)$ is non–degenerate.*

Proof. For the first part, note that by defining property (3) of the Shapovalov form, we have:

$$(\mu(h)u, v) = (u, \nu(h)v)$$

for every $h \in \mathfrak{h}$. If $\mu \neq \nu$ then there exists $h \in \mathfrak{h}$ such that $\mu(h) - \nu(h) \neq 0$ and hence $(u, v) = 0$.

For the second part, let us begin by assuming that $(.,.)$ is singular, i.e, there exists $u \in \mathrm{Rad}(.,.)$ and let us assume u is of maximal weight among vectors in $\mathrm{Rad}(.,.)$. Then for any $v \in M_\lambda$ we have

$$(e_i u, v) = (u, f_i v) = 0$$

which combined with the assumption on the weight of u implies that $e_i u = 0$ for each i. Thus the submodule generated by u is proper submodule of M_λ proving that it is not irreducible.

Conversely if M_λ is not irreducible, then it contains a highest weight vector $u \in M_\lambda[\lambda - \beta]$ for some $\beta \in Q_+ \setminus \{0\}$. Thus we have

$$(u, f_i v) = (e_i u, v) = 0$$

and since every vector other than $\mathbf{1}_\lambda$ is of the form $f_i v$ we have $(u, v) = 0$ for every $v \notin M_\lambda[\lambda]$. This combined with the fact that $\beta \neq 0$ proves that u belongs to the radical of $(.,.)$. $\square$

Theorem. *M_λ is irreducible for generic λ. More precisely M_λ is irreducible if, and only if $\langle \lambda+\rho, \alpha\rangle \neq n\dfrac{\langle\alpha,\alpha\rangle}{2}$ for every $\alpha\in R_+$ and $n\geq 1$.*

Proof. For $\gamma\in Q_+$ let $F_\gamma(\lambda)$ be the determinant of $(.,.)|_{M_\lambda[\lambda-\gamma]}$. The proof of this theorem is based on the following two claims:
Claim 1 $F_\gamma(\lambda)$ is a product of linear polynomials of the form:

$$\langle\lambda+\rho,\beta\rangle-\frac{1}{2}\langle\beta,\beta\rangle$$

for $\beta\in Q_+\setminus\{0\}$.

Claim 2 If $\langle\lambda+\rho,\alpha\rangle\neq\frac{n}{2}\langle\alpha,\alpha\rangle$ for all $\alpha\in R_+$ and positive integer n, then $F_\gamma(\lambda)\neq 0$ for every $\gamma\in Q_+$.

Given these two claims, the proof of the theorem follows. Let us begin by proving Claim 1.
Proof of Claim 1: Let us assume that $F_\gamma(\lambda)=0$ for some $\gamma\in Q_+$. Then by previous proposition M_λ is reducible and hence contains a copy of $M_{\lambda-\beta}$ for some $\beta\in Q_+\setminus\{0\}$. Using the fact that the Casimir operator $C\in U(\mathfrak{g})$ acts by $\langle\lambda+2\rho,\lambda\rangle$ on M_λ we get:

$$\langle\lambda-\beta+2\rho,\lambda-\beta\rangle=\langle\lambda+2\rho,\lambda\rangle$$

which implies that $\langle\lambda+\rho,\beta\rangle=\frac{1}{2}\langle\beta,\beta\rangle$ and the claim follows.

Proof of Claim 2: In view of claim 1, we can write

$$F_\gamma(\lambda)=\prod_{\beta\in S_\gamma(\lambda)}(\langle\lambda,\beta\rangle-\langle\rho+\beta/2,\beta\rangle)$$

where $S_\gamma(\lambda)$ is some finite subset of Q_+. We will show that $S_\gamma(\lambda)\subset\mathbb{N}R_+$. For this it suffices to show that the leading term of $F_\gamma(\lambda)$ is of the form $\prod\langle\lambda,n\alpha\rangle$.

Let $\alpha^{(1)},\cdots,\alpha^{(N)}$ be an ordering on R_+. For $\mathbf{n}\in\mathbb{N}^N$ define $a(\mathbf{n}):=f_{\alpha^{(1)}}^{n_1}\cdots f_{\alpha^{(N)}}^{n_N}.\mathbf{1}_\lambda\in M_\lambda$. Further let $a(\mathbf{n},\mathbf{m})=(a(\mathbf{n}),a(\mathbf{m})$. Then

$$\det(a(\mathbf{n},\mathbf{m}))=\sum_\sigma(-1)^\sigma\prod_{\mathbf{n}}a(\mathbf{n},\sigma(\mathbf{n}))$$

and it is easy to see that the term corresponding to $\sigma=1$ has the largest degree and $a(\mathbf{n},\mathbf{n})=\prod\langle\lambda,\alpha^{(j)}\rangle^{n_j}$. □

1.4. **Expectation value.** Let V be a finite–dimensional $\mathfrak{g}$–module. For any $\lambda,\mu\in\mathfrak{h}^*$ define:

$$\langle\bullet\rangle:\mathrm{Hom}_{\mathfrak{g}}(M_\lambda,M_\mu\otimes V)\to V[\lambda-\mu]$$

by $\langle\Phi\rangle=\langle\mathbf{1}_\mu^*,\Phi(\mathbf{1}_\lambda)\rangle$.

Proposition. *Let γ be a weight of V. Then the expectation value map*

$$\langle\bullet\rangle:\mathrm{Hom}_{\mathfrak{g}}(M_\lambda,M_{\lambda-\gamma}\otimes V)\to V[\gamma]$$

is an isomorphism for (a) generic λ (so that $M_{\lambda-\gamma}$ is irreducible) or (b) dominant integral λ which is sufficiently large (compared to γ).

PROOF. It is clear that $\mathrm{Hom}_{\mathfrak{g}}(M_\lambda, M_{\lambda-\gamma}\otimes V)$ is isomorphic to the space of highest weight vectors of highest weight λ in $M_{\lambda-\gamma}\otimes V$, which identifies the expectation value homomorphism with the projection to $\mu = 0$ component:

$$\left(\bigoplus_{\substack{\mu\in Q_+\\ \mu+\gamma\in P(V)}} M_{\lambda-\gamma}[\lambda-\gamma-\mu]\otimes V[\gamma+\mu]\right)^{\mathfrak{n}_+} \to V[\gamma]$$

where $P(V)$ is the set of weights of V.

This projection map is an isomorphism precisely when $M_{\lambda-\gamma}[\lambda-\gamma-\mu]$ does not contain singular vectors for $\mu \in Q_+$, $\mu \neq 0$ such that $\mu+\gamma \in P(V)$. Since V is finite–dimensional , there are only finitely many of such μ and for either λ generic, or λ sufficiently large, one can assume that this condition is true. $\square$

Thus we have the following equation for $\Phi : M_\lambda \to M_\mu \otimes V$:

$$\Phi(\mathbf{1}_\lambda) = \mathbf{1}_\mu \otimes \langle\Phi\rangle + \sum a_i \otimes b_i \tag{1.1}$$

where $\mathrm{wt}(a_i) < \mu$ and $\mathrm{wt}(b_i) > \lambda - \mu$.

For a weight vector $v \in V$ let Φ_λ^v denote the intertwiner such that $\langle\Phi_\lambda^v\rangle = v$:

$$\Phi_\lambda^v : M_\lambda \to M_{\lambda-\mathrm{wt}(v)} \otimes V$$

1.5. **Fusion operator.** Let V, W be two finite–dimensional $\mathfrak{g}$–modules and $v \in V$, $w \in W$ be two weight vectors. Fix λ generic, or sufficiently large dominant integral weight and consider the following homomorphism:

$$v\otimes w \mapsto \left\langle \Phi^v_{\lambda-\mathrm{wt}(v)} \otimes 1_W \circ \Phi^w_\lambda \right\rangle \in (V\otimes W)[\mathrm{wt}(v)+\mathrm{wt}(w)]$$

which we denote by $J_{VW}(\lambda)$, called the fusion operator. The following properties of the fusion operator are immediate from definitions and (1.1)

Proposition. (1) *$J_{VW}(\lambda)$ is $\mathfrak{h}$–module homomorphism.*
(2) *$J_{VW}(\lambda)$ is lower triangular with 1's on the diagonal.*
(3) *$J_{VW}(\lambda)$ is a rational function of λ.*

Example. Let $\mathfrak{g} = \mathfrak{sl}_2$ and L_m be the $m+1$–dimensional $\mathfrak{g}$–module with basis $\{v_0, \cdots, v_m\}$ and $\mathfrak{sl}_2$ action given by:

$$ev_i = iv_{i-1} \quad fv_i = (m-i)v_{i+1} \quad hv_i = (m-2i)v_i$$

Let $\xi_i(\mu) \in M_\mu \otimes L_m$ be given by:

$$\xi_i(\mu) = \sum_{k=0}^{i} \frac{(-1)^k}{k!} \frac{\begin{pmatrix} i \\ k \end{pmatrix}}{\begin{pmatrix} \mu \\ k \end{pmatrix}} (f^k \mathbf{1}_\mu) \otimes v_{i-k}$$

Then $e\xi_i(\mu) = 0$. Hence for each λ we have the following explicit expression for the intertwiner:

$$\Phi_\lambda^{v_i} : \mathbf{1}_\lambda \mapsto \xi_i(\lambda - m + 2i)$$

Thus fusion operator $J_{L_m,L_n}(\lambda)$ can be computed as:

$$J_{L_m,L_n}(\lambda)(v_i \otimes v_j) = \sum_{k=0}^{\min(j,m-i)} (-1)^k \begin{pmatrix} m-i \\ k \end{pmatrix} \frac{\begin{pmatrix} j \\ k \end{pmatrix}}{\begin{pmatrix} \lambda - n + 2j \\ k \end{pmatrix}} v_{i+k} \otimes v_{j-k}$$

1.6. **Dynamical twist equation.** Let us introduce the dynamical notation. For a function $F : \mathfrak{h}^* \to \mathrm{End}(V_1 \otimes \cdots \otimes V_n)$ we write $F(\lambda + h^j)$ for the function:

$$F(\lambda + h^j)(v_1 \otimes \cdots \otimes v_n) = F(\lambda + \mathrm{wt}(v_j))(v_1 \otimes \cdots \otimes v_n)$$

Proposition. *Let V, W, U be three finite–dimensional $\mathfrak{g}$–modules. Then we have:*

$$J_{V\otimes W,U}(\lambda) J_{VW}(\lambda - h^3) = J_{V,W\otimes U}(\lambda) J_{WU}(\lambda)$$

Proof. For weight vectors $v \in V, w \in W, u \in U$ we claim that both sides of the equation on $v \otimes w \otimes u$ are equal to:

$$\left\langle \left(\Phi^v_{\lambda-\mathrm{wt}(w)-\mathrm{wt}(u)} \otimes 1 \otimes 1\right) \circ \left(\Phi^w_{\lambda-\mathrm{wt}(u)} \otimes 1\right) \circ \Phi^u_\lambda \right\rangle$$

This assertion is proved by computing this expectation value in two different ways and using the following equation, which follows directly from the definitions:

$$\Phi^v_{\mu-\mathrm{wt}(w)} \otimes 1 \circ \Phi^w_\mu = \Phi_\mu^{J_{VW}(\mu)(v\otimes w)}$$

Using this we have:

$$\begin{aligned} &\left(\Phi^v_{\lambda-\mathrm{wt}(w)-\mathrm{wt}(u)} \otimes 1 \otimes 1\right) \circ \left(\Phi^w_{\lambda-\mathrm{wt}(u)} \otimes 1\right) \circ \Phi^u_\lambda \\ &= \left(\Phi^{J_{VW}(\lambda-\mathrm{wt}(u))(v\otimes w)}_{\lambda-\mathrm{wt}(u)} \otimes 1\right) \circ \Phi^u_\lambda = \Phi_\lambda^{\text{L.H.S.}} \end{aligned}$$

Similarly, we have:

$$\begin{aligned} &\left(\Phi^v_{\lambda-\mathrm{wt}(w)-\mathrm{wt}(u)} \otimes 1 \otimes 1\right) \circ \left(\Phi^w_{\lambda-\mathrm{wt}(u)} \otimes 1\right) \circ \Phi^u_\lambda \\ &= \left(\Phi^v_{\lambda-\mathrm{wt}(w)-\mathrm{wt}(u)} \otimes 1 \otimes 1\right) \circ \Phi_\lambda^{J_{W,U}(\lambda)(w\otimes u)} = \Phi_\lambda^{\text{R.H.S.}} \end{aligned}$$

□

Let $P : V \otimes W \to W \otimes V$ be the flip operator. Define:

$$J^{21}_{VW}(\lambda) := P_{12} J_{VW}(\lambda) P_{12} \in \mathrm{End}(W \otimes V)$$

$$J^{13}_{VU}(\lambda - h^2) := P_{23} J^{12}_{VU}(\lambda - h^3) P_{23} \in \mathrm{End}(V \otimes W \otimes U)$$

$$J^{23}_{WU}(\lambda - h^1) := P_{12} P_{23} J^{12}_{WU}(\lambda - h^3) P_{23} P_{12} \in \mathrm{End}(V \otimes W \otimes U)$$

As a consequence of Proposition 1.6 we also have the following equations:

$$J^{13,2}_{V\otimes U,W}(\lambda) J^{13}_{VU}(\lambda - h^2) = J^{1,23}_{V,W\otimes U}(\lambda) J^{32}_{U,W}(\lambda)$$

$$J^{32,1}_{W\otimes U,V}(\lambda) J^{23}_{WU}(\lambda - h^1) = J^{2,13}_{W,V\otimes U}(\lambda) J^{31}_{UV}(\lambda)$$

1.7. **Exchange operator.** Define $R_{VW}(\lambda) := J_{VW}(\lambda)^{-1} J^{21}_{WV}(\lambda) : V \otimes W \to V \otimes W$. The following theorem now follows from the results of the previous section:

Theorem. *$R(\lambda)$ satisfies the quantum dynamical Yang–Baxter equation:*

$$R^{12}(\lambda - h^3) R^{13}(\lambda) R^{23}(\lambda - h^1) = R^{23}(\lambda) R^{13}(\lambda - h^2) R^{12}(\lambda)$$

1.8. **ABRR equation.** For $\lambda \in \mathfrak{h}^*$ let us denote by $\overline{\lambda} \in \mathfrak{h}$ the element obtained via the isomorphism $\mathfrak{h}^* \to \mathfrak{h}$. Define:

$$\theta(\lambda) := \overline{\lambda} + \overline{\rho} - \frac{1}{2} \sum_i x_i^2 \in U(\mathfrak{h})$$

where $\{x_i\}$ is an orthonormal basis of $\mathfrak{h}$.

Theorem. *Let V, W be two finite–dimensional representations of $\mathfrak{g}$. Choose $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $\langle e_\alpha, e_{-\alpha} \rangle = 1$, e.g, $e_{-\alpha} = d_\alpha f_\alpha$. Then we have:*

$$[J_{VW}(\lambda), 1 \otimes \theta(\lambda)] = \sum_{\alpha \in R_+} (e_{-\alpha} \otimes e_\alpha) J_{VW}(\lambda) \tag{1.2}$$

Moreover $J_{VW}(\lambda) \in \mathrm{End}_{\mathfrak{h}}(V \otimes W)$ is the unique element satisfying (1.2) *of the form $1 + \sum_{\beta > 0} \phi_\beta \otimes \psi_\beta$ where $\phi_\beta \in \mathrm{End}(V)[-\beta]$ and $\psi_\beta \in \mathrm{End}(W)[\beta]$.*

Proof. Let us begin by proving the uniqueness of a solution of (1.2). Namely we are looking for a solution of the form $1 + N(\lambda)$ where $N(\lambda) \in \sum_{\beta > 0} \mathrm{End}(V)[-\beta] \otimes \mathrm{End}(W)[\beta]$. Rewriting (1.2) for $J(\lambda) = 1 + N(\lambda)$ we get:

$$(1 \otimes \mathrm{ad}(\theta(\lambda))) N(\lambda) = -\left(\sum_{\alpha \in R_+} (e_{-\alpha} \otimes e_\alpha) \right) (1 + N(\lambda))$$

We claim that for generic λ, the operator $\mathrm{ad}(\theta(\lambda)) : \mathrm{End}(W)[\gamma] \to \mathrm{End}(W)[\gamma]$ is invertible for each $\gamma > 0$. Assuming this we can rewrite the equation above:

$$N(\lambda) = -(1 \otimes \mathrm{ad}(\theta(\lambda))^{-1}) \left(\sum_{\alpha \in R_+} (e_{-\alpha} \otimes e_\alpha) \right) (1 + N(\lambda))$$

Thus we are reduced to proving that the operator A:

$$A(X) := -(1 \otimes \mathrm{ad}(\theta(\lambda))^{-1}) \left(\sum_{\alpha \in R_+} (e_{-\alpha} \otimes e_\alpha) \right) (1 + X)$$

has a unique fixed point in $\oplus_{\gamma > 0} \mathrm{End}(V)[-\gamma] \otimes \mathrm{End}(W)[\gamma]$. But this is clear by nilpotence of multiplication by $\sum f_\alpha \otimes e_\alpha$ because of the finite–dimensionality of V and W.

Thus it remains to prove that $\mathrm{ad}(\theta(\lambda))$ is invertible. An easy computation shows that:

$$\mathrm{ad}(\theta(\lambda))|_{\mathrm{Hom}(W[\gamma_1], W[\gamma_2])} = \left(\langle \lambda + \rho, \gamma_2 - \gamma_1 \rangle + \frac{1}{2} \left(\langle \gamma_1, \gamma_1 \rangle - \langle \gamma_2, \gamma_2 \rangle \right) \right) .1$$

which implies the claim since $\mathrm{End}(W)[\beta] = \oplus_\gamma \mathrm{Hom}(W[\gamma], W[\gamma+\beta])$.

Next we give a sketch of the proof that $J(\lambda)$ satisfies (1.2). The trick is to introduce a new operator

$$F(\lambda)(v \otimes w) := \left\langle \Phi^v_{\lambda - \mathrm{wt}(w)} \otimes 1 \circ (C \otimes 1) \circ \Phi^w_\lambda \right\rangle$$

where $C \in U\mathfrak{g}$ is the Casimir element. Clearly we have:

$$F(\lambda)(v \otimes w) = \langle \lambda - \mathrm{wt}(w), \lambda - \mathrm{wt}(w) + 2\rho\rangle \, J_{VW}(\lambda)(v \otimes w)$$

However computing $F(\lambda)(v \otimes w)$ using the definition of $C = \sum f_\alpha e_\alpha + e_\alpha f_\alpha + \sum x_i^2$ one obtains:

$$F(\lambda)(v \otimes w) = \left(-2 \left(\sum_{\alpha \in R_+} e_{-\alpha} \otimes e_\alpha \right) + \langle \lambda, \lambda + 2\rho \rangle \right.$$
$$\left. -2(1 \otimes (\overline{\lambda} + \overline{\rho})) + \left(1 \otimes \sum_i x_i^2 \right) \right) J_{VW}(\lambda)(v \otimes w)$$

Comparing the two calculations, we gets the desired equation. □

Definition. The universal fusion operator $J(\lambda)$ is the unique solution of (1.2) of the form:

$$1 + \sum_{\beta > 0} U(\mathfrak{n}_-)_{-\beta} \otimes U(\mathfrak{b}_+)_\beta$$

in a completion of $(U(\mathfrak{n}_-) \otimes U(\mathfrak{b}_+))^{\mathfrak{h}}$. Define the universal exchange operator $R(\lambda) := J(\lambda)^{-1} J^{21}(\lambda)$.

1.9. **The case of quantum groups.** Let $U_q\mathfrak{g}$ be the Drinfeld–Jimbo quantum group and let $\mathcal{R}$ be its R–matrix. Define:

$$\mathcal{R}_0 := \mathcal{R} q^{-\sum x_i \otimes x_i}$$

One can similarly construct the intertwiners, fusion operators in this setting. The exchange operator is defined by:

$$\mathcal{R}(\lambda) := J_q(\lambda)^{-1} \mathcal{R}^{21} J_q^{21}(\lambda)$$

which is a trigonometric solution (rational function of q^λ) of the quantum dynamical Yang–Baxter equation.

The ABRR equation takes the following form:

Theorem. *$J_q(\lambda)$ is the unique solution (unipotent of weight zero) of the following equation:*

$$J_q(\lambda)(1 \otimes q^{2\theta(\lambda)}) = \mathcal{R}_0^{21} (1 \otimes q^{2\theta(\lambda)}) J_q(\lambda) \tag{1.3}$$

The proof follows along the same lines of that of Theorem 1.8, with the exception that the role of Casimir element is played by the quantum Casimir element defined as $uq^{-2\rho}$ where $u = S(b_i)a_i$ is the Drinfeld element (if $\mathcal{R} = a_i \otimes b_i$).

Remark. Let us write $J_q(\lambda) = J(\lambda) + O(\hbar)$. Using the fact that $\mathcal{R}_0 = 1 + \hbar \sum_{\alpha>0} e_\alpha \otimes e_{-\alpha} + O(\hbar^2)$ one obtains (1.2) from (1.3) by taking coefficient of $\hbar$.

1.10. **Limits of the fusion operator.** One can use the ABRR equations (1.2) and (1.3) to compute the limits of $J(\lambda)$ as $\lambda \to \pm\infty$ in a suitable sense.

Theorem. (1) *For $q = 1$ (the classical case) we have the following:*

$$\lim_{\lambda \to \pm\infty} J(\lambda) = 1$$

where the limit $\lambda \to \pm\infty$ signifies that $|\lambda(h_i)| \to \infty$ for each $i \in I$.

(2) *For the quantum case we take the limits $q^\lambda \to \infty$ and $q^\lambda \to 0$ respectively.*

$$\lim_{q^\lambda \to \infty} J_q(\lambda) = 1$$

$$\lim_{q^\lambda \to 0} J_q(\lambda) = \mathcal{R}_0^{21}$$

Proof. We prove (2) only. The proof of (1) is same in spirit and in fact easier. Let us write $J = \sum_{\beta \geq 0} J^{(\beta)}$ and $\left(\mathcal{R}_0^{21}\right)^{-1} = \sum_{\beta \geq 0} S^{(\beta)}$. Here

$$J^{(\beta)}(\lambda), S^{(\beta)} \in (U_q \mathfrak{n}_-)[-\beta] \otimes (U_q \mathfrak{b}_+)[\beta]$$

Using (1.3) we have:

$$\begin{aligned} \lim_{q^\lambda \to \infty} J(\lambda) &= \lim_{q^\lambda \to \infty} (1 \otimes q^{-2\theta(\lambda)}) \left(\mathcal{R}_0^{21}\right)^{-1} J(\lambda)(1 \otimes q^{2\theta(\lambda)}) \\ &= \lim_{q^\lambda \to \infty} \sum_{\beta,\gamma \geq 0} q^{-2\langle \lambda+\rho, \beta+\gamma\rangle} \left((1 \otimes q^{x_i^2}) S^{(\gamma)} J^{(\beta)}(\lambda)(1 \otimes q^{-x_i^2}) \right) \\ &= 1 \end{aligned}$$

Similarly we have:

$$\begin{aligned} \lim_{q^\lambda \to 0} &= \lim_{q^\lambda \to 0} \mathcal{R}_0^{21} (1 \otimes q^{2\theta(\lambda)}) J(\lambda)(1 \otimes q^{-2\theta(\lambda)}) \\ &= \mathcal{R}_0^{21} \lim_{q^\lambda \to 0} \sum_{\beta \geq 0} q^{2\langle \lambda+\rho, \beta\rangle} \left((1 \otimes q^{-x_i^2}) J^{(\beta)}(\lambda)(1 \otimes q^{x_i^2}) \right) \\ &= \mathcal{R}_0^{21} \end{aligned}$$

□

2. Dynamical Weyl group

This section is aimed at defining certain operators $A_{w,V}(\lambda)$ depending on $\lambda \in \mathfrak{h}^*$, for each $w \in W$ and V an integrable $\mathfrak{g}$ (or $U_q\mathfrak{g}$) module. The main reference for this part is [4, §3–5]. [2]

We consider the quantum group $U_q(\mathfrak{g})$, for q a complex number, not a root of unity. We also fix a logarithm of q and have $q^2 = e^\hbar$. To have a uniform description, we allow $q = 1$ or $\hbar = 0$ however.

[2] based on the talk by Martina Balagovic

2.1. **Quantum Verma identities.** Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight and M_λ be the Verma module. Fix $w \in W$ and a reduced expression of $w = s_{i_1} \cdots s_{i_l}$ and define:

$$\alpha^{(j)} := s_{i_l} \cdots s_{i_{j+1}} \alpha_{i_j}$$
$$n_j := 2 \frac{\langle \lambda + \rho, \alpha^{(j)} \rangle}{\langle \alpha^{(j)}, \alpha^{(j)} \rangle}$$

Lemma. *With the preceding notations, we have*

(1) *The set* $\{(n_j, d_j) : j = 1, \cdots, l\}$ *is independent of the reduced expression of* w.
(2) *The element* $f_{i_1}^{n_1} \cdots f_{i_l}^{n_l} \in U_q(\mathfrak{n}_-)$ *is independent of the reduced expression of* w.
(3) *The vector* $f_{i_1}^{(n_1)} \cdots f_{i_l}^{(n_l)} \mathbf{1}_\lambda \in M_\lambda$ *is a singular vector of* $M_\lambda[w \cdot \lambda]$, *which is independent of the reduced expression of* w.

Notation: Recall the shifted action of W on $\mathfrak{h}^*$ written as $w \cdot \lambda$ is given by:

$$w \cdot \lambda = w(\lambda + \rho) - \rho$$

We will denote by $\mathbf{1}^\lambda_{w\cdot\lambda}$ the singular vector of $M_\lambda[w \cdot \lambda]$ obtained in (3) of Lemma above.

Proof. One can prove using an easy induction argument on $l(w)$ that the vector $\mathbf{1}^\lambda_{w\cdot\lambda} \in M_\lambda[w \cdot \lambda]$ and is a singular vector. To prove the independence from the choice of a reduced expression, using Tits' Lemma, we can assume that two reduced expressions differ by a braid move, thus it suffices to prove the claim for rank 2 cases: $A_1 \times A_1$, A_2, B_2 and G_2 and $w = w_0$ is the longest element.

In this case the set $\{\alpha^{(j)} : 1 \leq j \leq l\}$ is the set of positive roots, each with multiplicity one. This proves (1). For the rest a simple calculation implies that we only need to check the following equalities in the respective cases (for every $a, b \geq 0$):

$(A_1 \times A_1)$
$$f_1^a f_2^b = f_2^b f_1^a$$

(A_2)
$$f_1^a f_2^{a+b} f_1^b = f_2^b f_1^{a+b} f_2^b$$

(B_2)
$$f_1^a f_2^{a+b} f_1^{a+2b} f_2^b = f_2^b f_1^{a+2b} f_2^{a+b} f_1^a$$

(G_2)
$$f_1^a f_2^{a+b} f_1^{2a+3b} f_2^{a+2b} f_1^{a+3b} f_2^b = f_2^b f_1^{a+3b} f_2^{a+2b} f_1^{2a+3b} f_2^{a+b} f_1^a$$

Of course it is possible to prove these from the Serre relations. However a more conceptual proof goes as follows. First we know that both sides applied to $\mathbf{1}_\lambda$ are singular vectors of $M_\lambda[w\cdot\lambda]$. We claim that the space of singular vectors in $M_\lambda[w\cdot\lambda]$ is 1–dimensional. Assuming this, we get that both sides of the equations are related by a non–zero scalar. Then one can project $U_q\mathfrak{n}_-$ to the q–commutative algebra generated by $\{f_i : i = 1, 2\}$ with relation $f_i f_j = q_i^{a_{ij}} f_j f_i$ which implies that the scalar is in fact equal to 1.

Let $k = \dim\left(M_\lambda[w\cdot\lambda]^{U_q\mathfrak{n}_+}\right)$. Since λ is dominant integral, $M_{w\cdot\lambda}$ is irreducible and hence we have $M_{w\cdot\lambda}^{\oplus k} \subset M_\lambda$. Comparing dimensions of the weight spaces on the both sides, we obtain that $k=1$. □

2.2. **Dynamical Weyl group for $\mathfrak{sl}_2$.** In this paragraph $\mathfrak{g} = \mathfrak{sl}_2$ and we identify $\mathfrak{h}^*$ with $\mathbb{C}$ via $z\alpha/2 \leftrightarrow z$. The Weyl group in this case is $W = \langle 1, s | s^2 = 1\rangle$. Under the identification $\mathfrak{h}^* = \mathbb{C}$, the action of s is by multiplication by (-1).

For $\lambda \in \mathbb{N}$ and V a finite–dimensional $U_q\mathfrak{sl}_2$–module, $v \in V[\mu]$ we have the intertwiner:

$$\Phi^v_\lambda : M_\lambda \to M_{\lambda-\mu}\otimes V$$

Proposition. *The intertwiner Φ^v_λ restricts to a homomorphism:*

$$\begin{tikzcd}
M_\lambda \arrow[r, "\Phi^v_\lambda"] & M_{\lambda-\mu}\otimes V \\
M_{s\cdot\lambda} \arrow[u, hook] \arrow[r] & M_{s\cdot(\lambda-\mu)}\otimes V \arrow[u, hook]
\end{tikzcd}$$

Proof. Let us begin by writing:

$$\Phi^v_\lambda \mathbf{1}_\lambda = \mathbf{1}_{\lambda-\mu}\otimes v + \sum_{p>0} a_p \otimes v_p$$

where $a_p \in M_\lambda[\lambda-\mu-p]$ and $v_p \in V[\mu+p]$. Applying $f^{(\lambda+1)}$ to both sides we can write:

$$\Phi \mathbf{1}^\lambda_{s\cdot\lambda} = f^{(m)}\mathbf{1}_{\lambda-\mu}\otimes v' + \sum_{p>m} a'_p \otimes v'_p$$

Using the fact that $\mathbf{1}^\lambda_{s\cdot\lambda}$ is singular, we obtain that $f^{(m)}\mathbf{1}_{\lambda-\mu}$ is a singular vector and hence $m = \lambda-\mu+1$ and we are done. □

Define $A_{s,V}(\lambda) : V[\nu] \to V[s(\nu)]$ by:

$$\Phi^v_\lambda \mathbf{1}^\lambda_{s\cdot\lambda} = \mathbf{1}^{\lambda-\nu}_{s\cdot\lambda-\nu}\otimes A_{s,V}(v) + \cdots$$

2.3. **Computation of $A_{s,V}(\lambda)$.** We will need the following notational set up. Recall that for each $n \geq 0$ there is a unique irreducible $U_q\mathfrak{sl}_2$ module of dimension $n+1$, L_n. The action of $U_q\mathfrak{sl}_2$ is given on a basis $\{v_0, \cdots, v_n\}$ by:

$$Ev_j = [n-j+1]v_{j-1} \qquad Fv_j = [j+1]v_{j-1} \qquad Kv_j = q^{n-2j}v_j$$

Let M_λ be the Verma module. To describe the action of $U_q\mathfrak{sl}_2$ we fix a basis $\{m_r(\lambda) := f^{(r)}\mathbf{1}_\lambda : r \geq 0\}$

$$Em_r(\lambda) = [\lambda-r+1]m_{r-1}(\lambda) \quad Fm_r(\lambda) = [r+1]m_{r+1}(\lambda) \quad Km_r(\lambda) = q^{\lambda-2r}m_r(\lambda)$$

Define:

$$c_{k;r}^{(n)}(\mu) := (-1)^r q^{-r(n-2k+r+1)} \frac{\begin{bmatrix} n-k+r \\ r \end{bmatrix}_q}{\begin{bmatrix} \mu \\ r \end{bmatrix}_q}$$

$$\xi_k^{(n)}(\mu) = \sum_{r=0}^{k} c_{k;r}^{(n)}(\mu) m_r(\mu) \otimes v_{k-r} \in M_\mu \otimes L_n$$

Then it is easy to see that $\xi_k^{(n)}(\mu)$ is a singular vector. Hence we have:

$$\Phi_\lambda^{v_k} \mathbf{1}_\lambda = \xi_k^{(n)}(\lambda - n + 2k)$$

In order to compute $\Phi_\lambda^{v_k} \mathbf{1}_{s\cdot\lambda}^{\lambda}$ we will need the following computation, easily proved by induction:

$$\Delta(f^{(\lambda+1)}) = \sum_{r=0}^{\lambda+1} q^{r(\lambda+1-r)} f^{(\lambda+1-r)} K^{-r} \otimes f^{(r)}$$

Thus we have (in the following computations $\mu = \lambda - n + 2k$):

$$\begin{aligned} \Phi_\lambda^{v_k} \mathbf{1}_{s\cdot\lambda}^{\lambda} &= \Delta(f^{(\lambda+1)}) \xi_k^{(n)}(\mu) \\ &= \sum_{\substack{0 \le r \le \lambda+1 \\ 0 \le t \le k}} c_{k;t}^{(n)}(\mu) q^{r(n-2k+1-r+2t)} \begin{bmatrix} \lambda+1-r+t \\ t \end{bmatrix}_q \begin{bmatrix} k-t+r \\ r \end{bmatrix}_q m_{\lambda+1-r+t}(\mu) \otimes v_{k-t+r} \end{aligned}$$

Let us write $A_{s,L_n}(\lambda)(v_k) = A_n^k(\lambda) v_{n-k}$. Then taking coefficient of $m_{\mu+1}(\mu)$ from the above summation, we get:

$$A_n^k(\lambda) = q^{n-2k} \sum_{t=0}^{k} (-1)^t \begin{bmatrix} n-k+t \\ k \end{bmatrix}_q \begin{bmatrix} k \\ t \end{bmatrix}_q \frac{[\lambda - n + 2k + 1]}{[\lambda - n + 2k - s + 1]}$$

Lemma. *We have the following equality for each $n, k \in \mathbb{N}$, $k \le n$*

$$\sum_{t=1}^{k+1} (-1)^{t+1} \begin{bmatrix} n-t+1 \\ k \end{bmatrix}_q \begin{bmatrix} k \\ t-1 \end{bmatrix}_q \frac{1}{[\lambda - n + k + t]} = \frac{\prod_{j=1}^{k} [\lambda + j + 1]}{\prod_{j=1}^{k+1} [\lambda - n + k + j]}$$

The proof is an easy argument using partial fractions.

This lemma directly implies that

$$A_n^k(\lambda) = (-1)^k q^{n-2k} \prod_{j=1}^{k} \frac{[\lambda + j + 1]}{[\lambda - n + k + j]} \tag{2.1}$$

2.4. **Dynamical Weyl group for arbitrary $\mathfrak{g}$.** Let V be an integrable $U_q\mathfrak{g}$–module. For each $\nu \in P(V)$ and s_i a simple reflection define $A_{s_i,V}(\lambda) : V[\nu] \to V[s_i\nu]$ by considering the $(U_{q_i}\mathfrak{sl}_2)^{(i)}$ corresponding the root α_i:

$$A_{s_i,V}(\lambda) := A_{s,V}(\lambda(h_i))$$

Proposition. *Let $w = s_{i_1} \cdots s_{i_l}$ be a reduced expression for $w \in W$. Define:*

$$A_{w,V}(\lambda) := A_{s_{i_1},V}((s_{i_2} \cdots s_{i_l}) \cdot \lambda) \cdots A_{s_{i_{l-1}},V}(s_{i_l} \cdot \lambda) A_{s_{i_l},V}(\lambda)$$

Then $A_{w,V}(\lambda) : V[\nu] \to V[w\nu]$ and has the following form:

$$\Phi^v_\lambda \mathbf{1}^\lambda_{w\cdot\lambda} = \mathbf{1}^{\lambda-\nu}_{w\cdot\lambda-\nu} \otimes A_{w,V}(\lambda)v + \cdots$$

and hence it is independent of the choice of the reduced expression.

Proof is an easy induction on $l(w)$.

Corollary. (1) *$A_{w,V}(\lambda) : V[\nu] \to V[w\nu]$ is invertible, rational function of q^λ (or λ if $q = 1$).*

(2) *Let B_W be the braid group of type $\mathfrak{g}$. For each $w \in B_W$ we can define $A_{w,V}(\lambda)$ by*

$$A_{T_i^{-1},V}(\lambda) := A_{s_i,V}(s_i \cdot \lambda)^{-1}$$

(3) *If $l(w_1w_2) = l(w_1) + l(w_2)$ then we have:*

$$A_{w_1w_2,V}(\lambda) = A_{w_1,V}(w_2 \cdot \lambda) A_{w_2,V}(\lambda) \tag{2.2}$$

2.5. **Relation with the fusion operator.**

Proposition. *For U, V two integrable $U_q\mathfrak{g}$–modules we have:*

$$A_{w,U\otimes V}(\lambda) J_{UV}(\lambda) = J_{UV}(w \cdot \lambda) A^{(2)}_{w,V}(\lambda) A^{(1)}_{w,U}(\lambda - h^2) \tag{2.3}$$

Proof. We compute both sides on $u \otimes v$. It is clear that the value of left–hand side on $u \otimes v$ is the expectation value of the bottom horizontal arrow of the following diagram:

$$\begin{tikzcd}
M_\lambda \arrow[r, "\Phi^v_\lambda"] & M_{\lambda-\mathrm{wt}(v)} \otimes V \arrow[r, "\Phi^u_{\lambda-\mathrm{wt}(v)}\otimes 1"] & M_{\lambda-\mathrm{wt}(u)-\mathrm{wt}(v)} \otimes U \otimes V \\
M_{w\cdot\lambda} \arrow[u, hook] \arrow[rr] & & M_{w\cdot(\lambda-\mathrm{wt}\, u-\mathrm{wt}(v))} \otimes U \otimes V \arrow[u, hook]
\end{tikzcd}$$

Similarly the value of the right–hand side on $u \otimes v$ is given by the expectation value of the composition of bottom horizontal arrows in the following diagram:

$$\begin{tikzcd}
M_\lambda \arrow[r, "\Phi^v_\lambda"] & M_{\lambda-\mathrm{wt}(v)} \otimes V \arrow[r, "\Phi^u_{\lambda-\mathrm{wt}(v)}\otimes 1"] & M_{\lambda-\mathrm{wt}(u)-\mathrm{wt}(v)} \otimes U \otimes V \\
M_{w\cdot\lambda} \arrow[u, hook] \arrow[r] & M_{w\cdot(\lambda-\mathrm{wt}(v))} \otimes V \arrow[u, hook] \arrow[r] & M_{w\cdot(\lambda-\mathrm{wt}\, u-\mathrm{wt}(v))} \otimes U \otimes V \arrow[u, hook]
\end{tikzcd}$$

$\square$

2.6. **Application of Proposition 2.5: I.** We define a few more versions of the braid group action on meromorphic functions of $\lambda \in \mathfrak{h}^*$ taking values in V, an integrable $U_q\mathfrak{g}$–module.

Definition. Let $w \in B_W$, $V, V_1, \cdots, V_N$ be integrable $U_q\mathfrak{g}$–modules.

(1) *Shifted dynamical action* For a meromorphic function $f(\lambda) \in V$:

$$(w \circ f)(\lambda) = A_{w,V}(w^{-1} \cdot \lambda) f(w^{-1} \cdot \lambda)$$

(2) *Unshifted dynamical action* Define:

$$\mathcal{A}_{w,V}(\lambda) := A_{w,V}\left(-\lambda - \rho + \frac{1}{2}h\right)$$

here h stands for the dynamical notation introduced before. More explicitly $A\left(-\lambda - \rho + \frac{1}{2}h\right)(v) = A\left(-\lambda - \rho + \frac{1}{2}\,\mathrm{wt}(v)\right)v$. Finally we have the unshifted dynamical action of B_W on the space of meromorphic functions $f(\lambda) \in V$:

$$(w * f)(\lambda) = \mathcal{A}_{w,V}(w^{-1}\lambda) f(w^{-1}\lambda)$$

(3) *Shifted multicomponent dynamical action* Define

$$A_{w,V_1,\cdots,V_N}(\lambda) := A^{(N)}_{w,V_N}(\lambda) A^{(N-1)}_{w,V_{N-1}}(\lambda - h^N) \cdots A^{(1)}_{w,V_1}(\lambda - h^2 - \cdots - h^N)$$

$$\mathcal{A}_{w,V_1,\cdots,V_N} := A_{w,V_1,\cdots,V_N}\left(-\lambda - \rho + \frac{1}{2}\sum_{j=1}^{N} h^j\right)$$

Then the shifted multicomponent dynamical action of B_W on the space of meromorphic functions $f(\lambda) \in V_1 \otimes \cdots \otimes V_N$ is given by:

$$(w \bullet f)(\lambda) := A_{w,V_1,\cdots,V_N}(w^{-1} \cdot \lambda) f(w^{-1} \cdot \lambda)$$

(4) *Unshifted multicomponent dynamical action*

$$(w \diamond f)(\lambda) := \mathcal{A}_{w,V_1,\cdots,V_N}(w^{-1}\lambda) f(w^{-1}\lambda)$$

Similarly define the multicomponent versions of the fusion operators:

$$J^{1\cdots N}(\lambda) := J^{1,[2,N]}(\lambda) J^{2,[3,N]}(\lambda) \cdots J^{N-1,N}(\lambda)$$

$$\mathcal{J}^{1,\cdots,N}(\lambda) = J^{1,\cdots,N}\left(-\lambda - \rho + \frac{1}{2}\sum_{j=1}^{N} h^j\right)$$

We have the following important corollary of Proposition 2.5

Corollary. *The following relations hold as operators on the space of meromorphic functions $f(\lambda) \in V_1 \otimes \cdots \otimes V_N$*

(1)

$$J^{1,\cdots,N}(w\bullet) = (w\circ) J^{1,\cdots,N}$$

(2)

$$\mathcal{J}^{1,\cdots,N}(w\diamond) = (w*) \mathcal{J}^{1,\cdots,N}$$

The proof is essentially an iterative use of Proposition 2.5, except that the meaning of these equations needs a word of explanation. (1) means the following equality for each $f(\lambda)$:

$$J^{1,\cdots,N}(\lambda)A_{w,V_1,\cdots,V_N}(w^{-1}\cdot\lambda)f(w^{-1}\cdot\lambda)$$
$$= A_{w,V_1\otimes\cdots\otimes V_N}(w^{-1}\cdot\lambda)J^{1,\cdots,N}(w^{-1}\cdot\lambda)f(w^{-1}\cdot\lambda)$$

2.7. **Application of Proposition 2.5: II.** One can use Proposition 2.5 to carry out the computation of $A_{s,V}(\lambda)$ for $\mathfrak{g}=\mathfrak{sl}_2$ more conceptually (also see the calculation in section 2.3). The idea of the computation is as follows:

- Compute the dynamical operator for $V=L_1$ the standard 2–dimensional representation. This is easy step and the answer is given by:

$$A_{s,L_1}(\lambda)v_0 = qv_1 \qquad\qquad A_{s,L_1}(\lambda)(v_1) = -q^{-1}\frac{[\lambda+2]}{[\lambda+1]}v_0 \tag{2.4}$$

- Use the fact that L_{m+1} is a subrepresentation generated by the highest weight vector of $L_1\otimes L_m$ to get the recurrence relation among A_{s,L_m}, which determines it up to a scalar.
- Use the limits $\lambda\to\pm\infty$ to fix the scalar.

Let us denote by $A^k_m(\lambda)$ the coefficient:

$$A_{s,L_m}(\lambda)v_k = A^k_m(\lambda)v_{m-k}$$

Proposition 2.5 yields the following:

$$A_{s,L_1\otimes L_m}(\lambda)J_{L_1,L_m}(\lambda) = J_{L_1,L_m}(-\lambda)A^{(2)}_{s,L_m}(\lambda)A^{(1)}_{s,L_1}(\lambda-h^2)$$

which gives the following recurrence relation (in what follows the symbol $\equiv$ means equality up to a factor which is independent of λ).

$$A^0_{m+1}(\lambda) = A^0_m(\lambda)A^0_1(\lambda-m)$$
$$A^k_{m+1}(\lambda)A^{k-1}_{m-1}(\lambda) \equiv A^k_m(\lambda)A^0_1(\lambda-m+2k)A^0_1(\lambda-m+2k-2)$$

with base condition $A^0_1=1$ and

$$A^1_1(\lambda)\equiv\frac{[\lambda+2]}{[\lambda+1]}$$

Upon solving this system we obtain:

$$A^k_m(\lambda) = c_{m,k}\frac{\prod_{k=1}^k[\lambda+1+j]}{\prod_{j=1}^k[\lambda-m+k+j]} \tag{2.5}$$

2.8. **Limits.** In order to compute the scalar $c_{m,k}$ we take the limits $q^\lambda \to \infty, 0$ respectively. From the expression of $A^k_m(\lambda)$ it is clear that these limits exist.

$$A^+_V := \lim_{q^\lambda\to\infty} A_{s,V}(\lambda)$$
$$A^-_V := \lim_{q^\lambda\to 0} A_{s,V}(\lambda)$$

Then we get:

$$q^{-k(m-k+1)}A^+_{L_m}(v_k) = q^{k(m-k+1)}A^-_{L_m}(v_k) =: A^\infty_{L_m}(v_k)$$

Then it is clear that $c_{m,k}$ is given by $A^\infty_{L_m}(v_k) = c_{m,k}v_{m-k}$.

Proposition. $A^\infty_{L_m}(v_k) = (-1)^k q^{m-2k} v_{m-k}$

Proof. Recall that by Theorem 1.10 we have

$$\lim_{q^\lambda\to\infty} J_{UV}(\lambda) = 1 \qquad \lim_{q^\lambda\to 0} J_{UV}(\lambda) = \mathcal{R}^{21}_0$$

This implies by Proposition 2.5

$$A^+_{U\otimes V} = \mathcal{R}^{21}_0(A^+_U \otimes A^+_V)$$

Therefore if we define $A' = Aq^{\frac{h(h+2)}{4}}$ then we get:

$$A'_{U\otimes V} = \mathcal{R}^{21}(A'_U \otimes A'_V)$$

And hence the defining property of $\mathcal{R}$ implies that:

$$A'f = -q^{-2}eA' \qquad A'e = -q^2 fA'$$

and the same holds for A^∞ since they are proportional: $A^\infty|_{L_m} q^{m(m+2)/4} = A'|_{L_m}$. Now we can proceed as follows:

$$A^+_{L_1^{\otimes m}} v_0^{\otimes m} = q^m v_1^{\otimes m}$$

which implies that for L_m we have:

$$A^\infty(v_0) = q^m v_m$$

and the desired result follows form the commutation between $\{e, f\}$ and A^∞ given above. □

2.9. **Relation with quantum Weyl group.** Now we return to general set up of an arbitrary simple Lie algebra $\mathfrak{g}$. For any chamber in $\mathfrak{h}^*$, say C, we have the following notion of limit of the dynamical Weyl group operators:

$$A^C_{w,V} := \lim_{\substack{\lambda\to\infty \\ \text{in } C \text{ direction}}} A_{w,V}(\lambda)$$

The following equation is easy to verify. For $w = s_{i_1}\cdots s_{i_l}$ a reduced expression, we have:

$$A^C_{w,V} = A^{\epsilon_1(C)}_{s_{i_1},V}\cdots A^{\epsilon_l(C)}_{s_{i_l}.V}$$

where $\epsilon_j(C) = \text{sign}\left\langle C, \alpha^{(j)}\right\rangle$. In particular we have the following:

Proposition. *Let us define $A^{\pm}$ to be A^C for C dominant (and anti–dominant respectively). Then $A^{\pm}_{w,V}$ is independent of the choice of reduced expression. In particular $T_i \mapsto A^{\pm}_{s_i}$ extends to (two) group homomorphisms.*

Recall the definition of q–Weyl group operators:

$$\mathbb{S}_i := \exp_{q_i^{-1}}(q_i^{-1} e_i K_i^{-1}) \exp_{q_i^{-1}}(-f_i) \exp_{q_i^{-1}}(q e_i K_i) q_i^{h_i(h_i+1)/2}$$

where the q–exponential is defined by:

$$\exp_q(x) = \sum_{n\geq 0} q^{n(n-1)/2} \frac{x^n}{[n]!}$$

Then for $\mathfrak{sl}_2$ the following computation can be directly verified:

$$\mathbb{S} v_j = (-1)^{m-j} q^{(m-j)(j+1)} v_{m-j}$$

and hence we have the following equality of operators on finite–dimensional modules over $U_q\mathfrak{sl}_2$:

$$A^+_{L_m} = (-1)^m \mathbb{S} \qquad\qquad A^-_{L_m} = q^h \mathbb{S}^{-1}$$

which directly implies the following important corollary:

Corollary. *The elements $\{\mathbb{S}_i : i \in I\}$ satisfy the braid relations (of type $\mathfrak{g}$) in any integrable $U_q\mathfrak{g}$–module.*

Remark. It is easy to check that the results of §2 apply to arbitrary Kac–Moody algebra $\mathfrak{g}$ and locally finite $\mathfrak{h}$–diagonalizable modules V.

3. Affine Lie algebras

Now we consider the case of (untwisted) affine Lie algebras. We closely follow the conventions of [4, 7]. The main references for this part are [1, 7, 8]. [3]

3.1. **Loop realization.** Recall the notations from §1.1. In addition we set

$$m := \frac{(\theta,\theta)}{2} \qquad\qquad h^\vee := 1 + \rho(\theta^\vee)$$

Note that $m = 1$ for simply–laced cases (A, D, E); $m = 3$ for G_2 and $m = 2$ otherwise. $h^\vee$ is known as the dual Coxeter number.

Define $\mathfrak{g}[z, z^{-1}]$ to be the Lie algebra of Laurent polynomials with coefficients in $\mathfrak{g}$. Let $\widehat{\mathfrak{g}}$ be the central extension of $\mathfrak{g}[z, z^{-1}]$ given by the following 2–cocycle:

$$\omega(x(k), y(l)) = k\delta_{k+l,0}(x, y)$$

where we denote by $x(n) := x.z^n$ for $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$. It is easy to see that $\omega : \mathfrak{g}[z, z^{-1}] \times \mathfrak{g}[z, z^{-1}] \to \mathbb{C}$ satisfies the cocycle condition:

$$\omega([a, b], c) + \text{ cyclic} = 0 \text{ for every } a, b, c \in \mathfrak{g}[z, z^{-1}]$$

[3] BASED ON THE TALK BY SALVATORE STELLA

Hence the following bracket defines a Lie algebra structure on $\widehat{\mathfrak{g}} := \mathfrak{g}[z, z^{-1}] \oplus \mathbb{C}c$:

$$[x(k), y(l)] := [x, y](k + l) + mk\delta_{k+l,0}(x, y)c$$

and $\mathrm{ad}(c) \equiv 0$. Let ∂ denote the following derivation of $\widehat{\mathfrak{g}}$:

$$\partial(c) = 0 \qquad\qquad \partial(x(n)) = nx(n)$$

Let $\widetilde{\mathfrak{g}} := \widehat{\mathfrak{g}} \rtimes \mathbb{C}d$. That is, $\widehat{\mathfrak{g}}$ is a Lie subalgebra of $\widetilde{\mathfrak{g}}$ and $\mathrm{ad}(d)(\bullet) = \partial(\bullet)$. Define $\widehat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}c$ and $\widetilde{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$.

We extend the inner product on $\mathfrak{h}$ to one on $\widetilde{\mathfrak{h}}$ by declaring $(c, d) = 1/m$ and $(c, c) = (d, d) = 0 = (c, \mathfrak{h}) = (d, \mathfrak{h})$. This bilinear form defines an isomorphism $\widetilde{\nu} : \widetilde{\mathfrak{h}}^* \to \widetilde{\mathfrak{h}}$. Let $\delta, \Lambda_0 \in \widetilde{\mathfrak{h}}^*$ be two linear forms dual to the elements d and c respectively. A typical element of $\widetilde{\mathfrak{h}}^*$ is written as $\widetilde{\lambda} = \lambda + k\Lambda_0 + l\delta$. Moreover $\widetilde{\nu}$ is determined by the following:

$$\widetilde{\nu}|_{\mathfrak{h}} = \nu$$

$$\widetilde{\nu}(\Lambda_0) = md \qquad\qquad \widetilde{\nu}(\delta) = mc$$

We have the root space decomposition of $\widetilde{\mathfrak{g}}$ relative to $\widetilde{\mathfrak{h}}$:

$$\widetilde{g} = \widetilde{\mathfrak{h}} + \bigoplus_{\widetilde{\alpha} \in \widehat{R}} \widetilde{\mathfrak{g}}_{\widetilde{\alpha}}$$

where $\widehat{R} = \{\alpha + n\delta :$ either $\alpha \in R, n \in \mathbb{Z}$ or $\alpha = 0, n \in \mathbb{Z}^{\times}\}$. We choose the following base of $\widehat{R}$:

$$\widehat{\Delta} := \{\alpha_i : i \in I\} \cup \{-\theta + \delta =: \alpha_0\}$$

Then the set of coroots $\widehat{\Delta}^{\vee} \in \widetilde{\mathfrak{h}}$ is given by:

$$\widehat{\Delta}^{\vee} = \{h_i : i \in I\} \cup \left\{h_0 := c - \theta^{\vee}\right\}$$

In order to obtain the Kac–Moody presentation of $\widetilde{\mathfrak{g}}$ let us choose $e_\theta \in \mathfrak{g}_\theta$ and $f_\theta \in \mathfrak{g}_{-\theta}$ determined by the condition that $(e_\theta, f_\theta) = 1/m$. Define

$$e_0 := f_\theta(1) \qquad\qquad f_0 := e_\theta(-1)$$

Then clearly $[e_0, f_0] = -\theta^{\vee} + c = h_0$. The system of generators $\{h_i, e_i, f_i : i \in \{0\} \cup I\}$ (and d) give a Kac–Moody presentation of $\widehat{\mathfrak{g}}$ (respectively $\widetilde{\mathfrak{g}}$).

We have the similar notions of affine root lattice, weight lattice etc. Let us choose $\widehat{\rho} \in \widetilde{\mathfrak{h}}^*$ so that $\widehat{\rho}(h_i) = 1$ for each $i \in \{0\} \cup I$ as follows:

$$\widehat{\rho} = \rho + h^{\vee}\Lambda_0$$

A word of caution: The notations we have used are consistent with [4, 7], but differ slightly from [2]. In [2] the normalization of $(.,.)$ is chosen so that $(\theta, \theta) = 2$, i.e, $m = 1$.

3.2. **Affine Weyl group.** Let us denote by W_{aff}, the *affine Weyl group*, the group generated by reflections $\langle s_0, s_i : i \in I\rangle \subset GL(\widetilde{\mathfrak{h}}^*)$. Using $\widetilde{\nu}$ we also have an action of W_{aff} on $\widetilde{\mathfrak{h}}$. In order to understand this group better, we introduce the following element:

$$t_{\theta^\vee} := s_0 s_\theta$$

The following equations can be verified by direct computations:

$$t_{\theta^\vee}(\lambda + k\Lambda_0 + l\delta) = \lambda + mk\nu^{-1}(\theta^\vee) + k\Lambda_0 + \left(l - \lambda(\theta^\vee) - mk\frac{(\theta^\vee, \theta^\vee)}{2}\right)\delta$$

$$t_{\theta^\vee}(h + kd + lc) = h + k\theta^\vee + kd + \left(l - m(h, \theta^\vee) - mk\frac{(\theta^\vee, \theta^\vee)}{2}\right)c$$

Let us define $t_{\alpha^\vee}$ for any $\alpha^\vee \in Q^\vee$ as an operator on $\widetilde{\mathfrak{h}}$ by:

$$t_{\alpha^\vee}(h + kd + lc) = h + k\alpha^\vee + kd + \left(l - m(h, \alpha^\vee) - \frac{mk}{2}(\alpha^\vee, \alpha^\vee)\right)c \tag{3.1}$$

Then the following properties can be easily verified:

$$w t_{\alpha^\vee} w^{-1} = t_{w\alpha^\vee} \qquad\qquad t_{\alpha^\vee} t_{\beta^\vee} = t_{\alpha^\vee + \beta^\vee}$$

which together with the fact that $t_{\theta^\vee} \in W_{\mathrm{aff}}$ and that $\theta^\vee$ is a short root of $R^\vee$ implies that $t_\alpha^\vee \in W_{\mathrm{aff}}$ for every $\alpha^\vee \in Q^\vee$. We obtain the following theorem (using the fact that the action of W_{aff} on $\widetilde{\mathfrak{h}}$ is faithful):

Theorem. *The assignment $s_0 \mapsto (s_\theta, -\theta^\vee)$ extends to an isomorphism of groups $W_{\mathrm{aff}} \cong W \ltimes Q^\vee$.*

3.3. **Extended affine Weyl group.** It is easy to see that (3.1) defined for $\alpha^\vee \in P^\vee$ preserves the set of roots $\widehat{R}$. This allows us to define a larger group of symmetries:

$$W^e_{\mathrm{aff}} := W \ltimes P^\vee \tag{3.2}$$

together with a length function $l(w)$ = number of positive roots mapped to negative by w. Let us denote by $\Pi \subset W^e_{\mathrm{aff}}$ the subgroup of elements of length 0. Let V be the real affine hyperplane in $\widetilde{\mathfrak{h}}/\mathbb{C}c$ defined by $\delta = 1$, which we can identify with $\mathfrak{h}_\mathbb{R}$ via $h \mapsto \overline{h + d}$. A quick computation using (3.1) implies that the action of W_{aff} and W^e_{aff} descends to an affine linear action on $\mathfrak{h}$ given by:

$$t_{\alpha^\vee}(h) = h + \alpha^\vee \qquad\qquad W \text{ acts as usual}$$

The linear form $\alpha + n\delta \in \widehat{R}$ gets identified with an affine linear function on $\mathfrak{h}$ given by: $(\alpha + n\delta)(h) = \alpha(h) + n$. Let $\mathfrak{h}^{\text{aff-reg}}$ be the complement in $\mathfrak{h}$ of the affine root hyperplanes: $\mathfrak{h}_{\alpha+n\delta} := \{h : \alpha(n) = -n\}$.

Let us choose the alcove in $\mathfrak{h}_\mathbb{R}$ defined by

$$C := \{h \in \mathfrak{h}_\mathbb{R} : \alpha_i(h) > 0 \text{ for every } i \in \{0\} \cup I\}$$

In other words $C = \{h : \alpha_i(h) > 0 \text{ and } \theta(h) < 1\}$. It is clear that C is an open simplex in $\mathfrak{h}_\mathbb{R}$ and the walls of $\overline{C}$ are canonically labeled by $\{0\} \cup I$. The elements of

Π permute the walls of $\overline{C}$ and hence act as symmetries of the affine Dynkin diagram. The following theorem is given in [1, Chapter 6, §2].

Theorem. *The subgroup* Π *of* W^e_{aff} *is isomorphic to* $P^\vee/Q^\vee$. *There is a canonical bijection between the set* $\Pi \setminus \{1\}$ *and the set of minuscule coweights:*

$$\Pi \setminus \{1\} \leftrightarrow J := \{i \in I : \theta(\omega_i^\vee) = 1\}$$

Let $1 \neq \pi \in \Pi$ *correspond to* $i \in J$. *Then as an element of* W^e_{aff} *we have:*

$$\pi = t_{\omega_i^\vee} w_i w_0$$

where w_i *is the longest element of the root system obtained by deleting* i *from the Dynkin diagram of* $\mathfrak{g}$. *Finally we have a group isomorphism* $W^e_{\text{aff}} \cong \Pi \ltimes W_{\text{aff}}$.

3.4. **Affine braid group.** Using the length function on W^e_{aff} we can define *the extended affine braid group* denoted by B^e_{aff} as a group generated by $\{T_w : w \in W^e_{\text{aff}}\}$ subject to the relations:

$$T_u T_v = T_{uv} \text{ if } l(uv) = l(u) + l(v)$$

Corresponding to the realizations $W^e_{\text{aff}} = W \ltimes P^\vee$ and $W^e_{\text{aff}} = \Pi \ltimes W_{\text{aff}}$ we have two presentations of B^e_{aff}:

(1) B^e_{aff} is generated by $\{T_w, Y^\gamma : w \in W, \gamma \in P^\vee\}$ subject to

$$\begin{aligned} T_u T_v &= T_{uv} \text{ if } l(uv) = l(u) + l(v) \\ Y^\lambda Y^\mu &= Y^{\lambda+\mu} \text{ for every } \lambda, \mu \in P^\vee \\ T_i^{-1} Y^\lambda T_i^{-1} &= \begin{cases} Y^\lambda & \text{if } \quad \alpha_i(\lambda) = 0 \\ Y^{s_i(\lambda)} & \text{if } \quad \alpha_i(\lambda) = 1 \end{cases} \end{aligned}$$

(2) B^e_{aff} is generated by $\{U_\pi, T_w : w \in W_{\text{aff}}, \pi \in \Pi\}$ subject to

$$\begin{aligned} T_u T_v &= T_{uv} \text{ if } l(uv) = l(u) + l(v) \\ U_\pi U_{\pi'} &= U_{\pi+\pi'} \\ U_\pi T_{s_i} U_\pi^{-1} &= T_{s_{\pi(i)}} \end{aligned}$$

The affine braid group B_{aff} is a subgroup of B^e_{aff} generated by $\{T_w : w \in W_{\text{aff}}\}$.

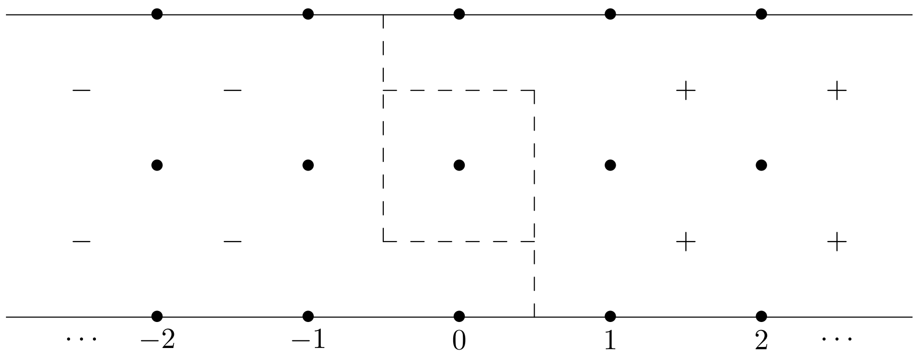
3.5. **Category $\mathcal{O}$ and finite–dimensional representations.** Since $\widetilde{\mathfrak{g}}$ is a Kac–Moody algebra one can define the notion of highest weight modules, category $\mathcal{O}$, Verma modules etc.

Definition. A representation V of $\widetilde{\mathfrak{g}}$ is said to be $\widetilde{\mathfrak{h}}$–diagonalizable if

$$V = \bigoplus_{\widetilde{\lambda} \in \widetilde{\mathfrak{h}}^*} V[\widetilde{\lambda}]$$

We say V is locally finite if for each $i \in I \cup \{0\}$, the operators e_i, f_i act locally nilpotently. The category $\mathcal{O}$ consists of finitely–generated $\widetilde{\mathfrak{h}}$–diagonalizable modules V such that there exist a finite collection $\{\widetilde{\lambda}_a : 1 \leq a \leq r\}$ such that $\widetilde{\mu} \in P(V)$ implies that $\widetilde{\mu} \leq \widetilde{\lambda}_a$ for some a.

FIGURE 1. Decomposition for $\mathcal{O}$



One defines the Verma module $M_{\widetilde{\lambda}}$ for each $\widetilde{\lambda} \in \widetilde{\mathfrak{h}}^*$ analogously and it is easy to show that $M_{\widetilde{\lambda}} \in \mathcal{O}$.

Now let us focus on another family of representations of $\widetilde{\mathfrak{g}}$ which do not belong to category $\mathcal{O}$. Let $\overline{V}$ be a finite–dimensional representation of $\widehat{\mathfrak{g}}$. One can prove easily that c acts by 0 on such representations. Define *loop representation* of $\widetilde{\mathfrak{g}}$ by $V := \overline{V}[z, z^{-1}]$ where $x(r)$ acts by $z^r x(r)$ and d acts by $z\frac{d}{dz}$. Then one has:

$$P(V) = \{\lambda + n\delta : \lambda \in P(\overline{V}) \subset \mathfrak{h}^*, n \in \mathbb{Z}\}$$

and V is $\widetilde{\mathfrak{h}}$–diagonalizable. Moreover V is also locally finite, but not in category $\mathcal{O}$.

Example. Let $\mathfrak{g} = \mathfrak{sl}_2$ and choose $a \in \mathbb{C}^\times$. Define $\overline{V} = L_n(a)$, a $\widehat{\mathfrak{g}}$–module with basis $\{v_0, \cdots, v_n\}$ and $\widehat{\mathfrak{g}}$–action given by:

$$e(r).v_i = a^r(n-i+1)v_{i-1} \qquad f(r).v_i = a^r(i+1)v_{i+1} \qquad h(r).v_i = a^r(n-2i)v_i$$

Then $\widetilde{\mathfrak{g}}$ action on the corresponding loop representation, V, with basis $\{v_i(s) : 0 \leq i \leq n, s \in \mathbb{Z}\}$ is given by:

$$e(r).v_i(s) = a^r(n-i+1)v_{i-1}(r+s) \qquad f(r).v_i(s) = a^r(i+1)v_{i+1}(r+s)$$
$$h(r).v_i(s) = a^r(n-2i)v_i(r+s) \qquad d.v_i(s) = sv_i(s)$$

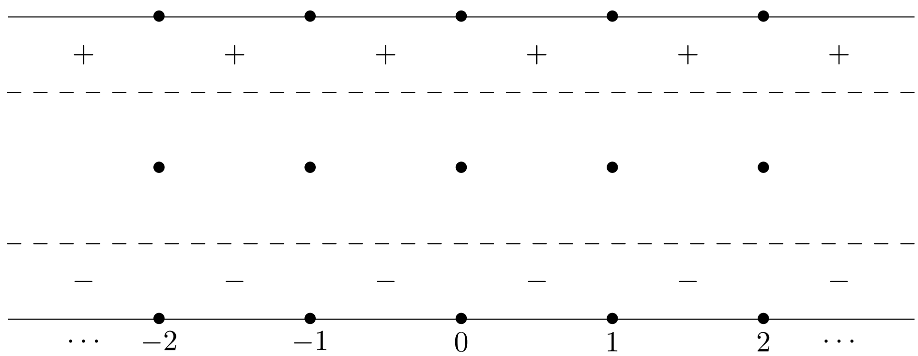
Then it is clear that the weights of V are not bounded from above.

Let us denote by $\mathrm{Rep}_{\mathrm{fd}}(\widetilde{\mathfrak{g}})$ the category of loop representations of $\widetilde{\mathfrak{g}}$ which is manifestly same as the category of finite–dimensional representations of $\mathfrak{g}[z, z^{-1}]$. We remark that the two notions essentially differ by choice of the triangular decomposition of $\widehat{\mathfrak{g}}$ (see Figures 1, 2 where each $\bullet$ on top row indicates $\mathfrak{n}_+$, on middle row indicates $\mathfrak{h}$ and on bottom row indicates $\mathfrak{n}_-$, spread over all $\mathbb{Z}$).

Finally we introduce the notion of shift of a module $\overline{V}$ by $b \in \mathbb{C}^\times$ as:

$$\overline{V}(b) := V/(z-b)V \in \mathrm{Rep}_{\mathrm{fd}}(\widehat{\mathfrak{g}})$$

FIGURE 2. Decomposition for $\mathrm{Rep}_{\mathrm{fd}}(\widetilde{\mathfrak{g}})$



4. Intertwiners and fusion operator for affine Lie algebras

In this section we consider the intertwiner operator between Verma module and a tensor product of Verma module with loop representation. The contents of this section carry over uniformly to the quantum affine algebras and we consider both the classical and quantum cases together. A brief discussion of quantum affine algebras is given later. [4]

4.1. **Completed tensor product I.** Let $\widetilde{\lambda} \in \widetilde{\mathfrak{h}}^*$ and let $M_{\widetilde{\lambda}}$ be the Verma module. Let V be a loop representation of $U_q\widetilde{\mathfrak{g}}$. We would like to have an analogue of Proposition 1.4 in this setting. Define the expectation value homomorphism as:

$$\langle \bullet \rangle : \mathrm{Hom}_{U_q\widetilde{\mathfrak{g}}}(M_{\widetilde{\lambda}}, M_{\widetilde{\mu}} \otimes V) \to V[\widetilde{\lambda} - \widetilde{\mu}]$$

In order for this map to be invertible, we need to consider the completed tensor product $\widehat{\otimes}$ defined as:

Definition. Let U and V be two diagonalizable $\widetilde{\mathfrak{h}}$ modules. Then:

$$U \widehat{\otimes} V := \bigoplus_{\widetilde{\mu}} \left(\prod_{\widetilde{\beta}} U[\widetilde{\mu} - \widetilde{\beta}] \otimes V[\widetilde{\beta}] \right)$$

Proposition. *The expectation value homomorphism* $\langle \bullet \rangle$ *is an isomorphism for generic* $\widetilde{\lambda}$ *and large dominant* $\widetilde{\lambda}$*:*

$$\langle \bullet \rangle : \mathrm{Hom}_{U_q\widetilde{\mathfrak{g}}}(M_{\widetilde{\lambda}}, M_{\widetilde{\mu}} \widehat{\otimes} V) \to V[\widetilde{\lambda} - \widetilde{\mu}]$$

Remark. The proposition would be false without the completion. For example, let $\mathfrak{g} = \mathfrak{sl}_2$ and $\widetilde{\lambda} = \lambda \in \mathbb{C}$. Take $\overline{V} = L_1$ to be the standard two dimensional representation. We claim that there are no singular vectors in $M_{\widetilde{\lambda}} \otimes V$. Assume the contrary that ξ is a singular vector and let $m \otimes v_i z^n$ be a non–zero term in ξ with largest n. Then $h(1)\xi$ has the term $(-1)^i m \otimes v_i z^{n+1}$ which is not cancelled by any other, which contradicts that $\widehat{\mathfrak{n}}_+ \xi = 0$.

[4] based on the talk by Andrea Appel

Thus it makes sense to consider an operator $\Phi^v_{\widetilde{\lambda}}$ whose expectation value in $v \in V$.

4.2. **Completed tensor product II.** We can similarly define the fusion operator $J_{UV}(\widetilde{\lambda}, z)$ for which we need the following completion of the tensor product.

Definition. Let $V_1, \cdots, V_N$ be loop representations of $U_q\widetilde{\mathfrak{g}}$. Define:

$$V_1 \vec{\otimes} \cdots \vec{\otimes} V_N := \left(\overline{V_1} \otimes \cdots \otimes \overline{V_N}\right)[[z_2/z_1, \cdots, z_N/z_{N-1}]][z_i, z_i^{-1} : i = 1, \cdots, N]$$

with natural $U_q\widetilde{\mathfrak{g}}$–module structure.

With this notation, we define the fusion operator $J_{UV}(\widetilde{\lambda}, z_1, z_2)$ whose value at $u \otimes v$ (for $u \in \overline{U}[\mathrm{wt}(u)]$ and $v \in \overline{V}[\mathrm{wt}(v)]$) is same as the expectation value of the following composition:

$$M_{\widetilde{\lambda}} \xrightarrow{\Phi^v_{\widetilde{\lambda}}} M_{\widetilde{\lambda}-\mathrm{wt}(v)} \widehat{\otimes} V \xrightarrow{\Phi^u_{\widetilde{\lambda}-\mathrm{wt}(v)} \otimes 1} M_{\widetilde{\lambda}-\mathrm{wt}(u)-\mathrm{wt}(v)} \widehat{\otimes} (U \vec{\otimes} V)$$

We claim that the fusion operator is in fact an element of $\mathrm{End}(\overline{U} \otimes \overline{V})[[z_2/z_1]]$ which allows us to write it as a formal series in $z = z_2/z_1$.

$$J_{UV}(\widetilde{\lambda}, z) = \sum_{n \geq 0} J_{UV,n} z^n$$

We remark that the usual properties of the fusion operator hold. That is, it is an invertible operator which is a rational function of $q^{\widetilde{\lambda}}$. Moreover Proposition 2.5 and Corollary 2.6 hold (where B_W in the statements is replaced by B_{aff}).

Remark. The following computation motivates the definition of $\vec{\otimes}$. Let $\Phi : M_{\widetilde{\lambda}} \to M_{\widetilde{\mu}} \widehat{\otimes} V$ be an intertwiner. We only consider weight spaces under d in what follows. Since $\widehat{\mathfrak{n}}_-$ has negative energy, we have ($l = \widetilde{\lambda}(d)$)

$$M_{\widetilde{\lambda}} = \bigoplus_{n \geq 0} M_{\widetilde{\lambda}}[l-n]$$

If $\langle \Phi \rangle = v.z^k$ for $v \in \overline{V}[\gamma]$ then Φ maps

$$\Phi^{vz^k}_{\widetilde{\lambda}} : M_{\widetilde{\lambda}}[l-n] \to \bigoplus_{t \geq 0} M_{\widetilde{\lambda}-\gamma-k\delta}[l-k-t] \otimes V[k+t-n]$$

Let us assume that $k = 0$ and let $u \in \overline{U}[\mu]$. Then the composition of $\Phi^u_{\widetilde{\lambda}-\gamma} \otimes 1 \circ \Phi^v_{\widetilde{\lambda}}$ maps:

$$M_{\widetilde{\lambda}}[l-n] \overset{\Phi^v_{\widetilde{\lambda}}}{\to} \prod_{t \geq 0} M_{\widetilde{\lambda}-\gamma}[l-t] \otimes V[t-n] \overset{\Phi^u_{\widetilde{\lambda}-\gamma}}{\to} \prod_{p,t \geq 0} M_{\widetilde{\lambda}-\gamma-\mu}[l-p] \otimes U[p-t] \otimes V[t-n]$$

The last term of the composition has z terms of the form $(z_2/z_1)^t . z_1^p z_2^{-n}$. This is the reason for considering $\vec{\otimes}$. Note that the fusion operator is the expectation value of this composition, and hence only considers the case $n = 0 = p$ for which we only get a power series in z_2/z_1 as claimed.

4.3. **Quantum affine algebras: two definitions.**

5. Trigonometric Knizhnik–Zamolodchikov equations

The aim of this section is to obtain a differential equation satisfied by the fusion operators for affine Lie algebra $\widetilde{\mathfrak{g}}$. Main references for this part are [2, 5].[5]

5.1. **Casimir operator.** Let $\mathfrak{g}$ be a Kac–Moody algebra. For each $\alpha \in R_+$ choose a basis $\{e_\alpha^{(s)}\}$ of $\mathfrak{g}_\alpha$ and the dual basis $\{e_{-\alpha}^{(s)}\}$ of $\mathfrak{g}_{-\alpha}$. The following construction of the Casimir element is taken from [7, §2.5]. Define:

$$C_0 := 2 \sum_{\alpha \in R_+} \sum_s e_{-\alpha}^{(s)} e_\alpha^{(s)} \in U\mathfrak{g}$$

Finally choose $\rho \in \mathfrak{h}^*$ such that $\rho(h_i) = 1$ for each i.

Proposition. *For $x \in \mathfrak{g}_\alpha$ we have:*

$$[C_0, x] = -x\left(2\langle \rho, \alpha\rangle + 2\nu(\alpha) + \langle \alpha, \alpha\rangle\right)$$

where $\alpha \in \{0\} \cup R$.

Proof. The assertion is clear for $\alpha = 0$ since C_0 is of weight 0. We claim that if the assertion holds for $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$ then it holds for $[x, y] \in \mathfrak{g}_{\alpha+\beta}$. To prove this we have:

$$\begin{aligned}
[C_0, [x, y]] &= [x, [C_0, y]] + [[C_0, x], y] \\
&= -[x, y]\left(2\langle \rho, \alpha + \beta\rangle + \langle \alpha, \alpha\rangle + \langle \beta, \beta\rangle\right) - 2\left([x, y\nu(\beta)] + [x\nu(\alpha), y]\right) \\
&= -[x, y]\left(2\langle \rho, \alpha + \beta\rangle + \langle \alpha + \beta, \alpha + \beta\rangle + 2\nu(\alpha + \beta)\right)
\end{aligned}$$

Thus it suffices to prove the proposition for $x \in \mathfrak{g}_{\alpha_i}$, i.e, $x = e_i$ (the proof for $x = f_i$ is similar). In order to carry out the computation, we need the following claim:
Claim: If $z \in \mathfrak{g}_{\beta-\alpha}$ where $\alpha, \beta \in \mathfrak{h}^*$ such that $\beta - \alpha \in R$, then we have:

$$\sum_r e_{-\alpha}^{(r)} \otimes [z, e_\alpha^{(r)}] = \sum_s [e_{-\beta}^{(s)}, z] \otimes e_\beta^{(s)}$$

Assuming this claim, we have:

$$\begin{aligned}
[C_0, e_i] &= 2 \sum_{\alpha \in R_+} \sum_j [e_{-\alpha}^{(j)}, e_i] e_\alpha^{(j)} + e_{-\alpha}^{(j)} [e_\alpha^{(j)}, e_i] \\
&= 2 \sum_j [e_{-\alpha_i}^{(j)}, e_i] e_{\alpha_i}^{(j)} + 2 \sum_{\alpha \in R_+ \setminus \{\alpha_i\}} \left([e_{-\alpha}^{(s)}, e_i] e_\alpha^{(s)} + e_{-\alpha+\alpha_i}^{(r)} [e_{\alpha-\alpha_i}^{(r)}, e_i]\right) \\
&= 2 \sum_j [e_{-\alpha}^{(j)}, e_i] e_{\alpha_i}^{(j)}
\end{aligned}$$

The last term can be easily seen to be equal to $-2\nu(\alpha_i)e_i$ which is same as $-e_i(2\nu(\alpha_i) + 2\langle \alpha_i, \alpha_i\rangle)$ as claimed in the proposition (since $2\langle \rho, \alpha_i\rangle = \langle \alpha_i, \alpha_i\rangle$).

[5] based on the talk by Sachin Gautam

Proof of the claim: Let us pair both sides of the required equation with $e \otimes e' \in \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\beta}$. Then by invariance and non–degeneracy of $\langle .,. \rangle$ we have:

$$\begin{aligned}\text{L.H.S.} &= \sum_r \left\langle e_{-\alpha}^{(r)}, e \right\rangle \left\langle [z, e_\alpha^{(r)}], e' \right\rangle \\ &= \sum_r \left\langle e_{-\alpha}^{(r)}, e \right\rangle \left\langle [e', z], e_\alpha^{(r)} \right\rangle \\ &= \left\langle e, [e', z] \right\rangle\end{aligned}$$

Similarly we get that the right–hand side is same as $\langle [x, e], e' \rangle$ and we are done. □

Corollary. *Define $C = C_0 + 2\nu(\rho) + \sum_i x_i^2$, where $\{x_i\}$ is an orthonormal basis of $\mathfrak{h}$. Then we have:*

$$[C, x] = 0 \textit{ for every } x \in \mathfrak{g}$$

Proof. In view of the proposition above, we only need to show that for every $x \in \mathfrak{g}_\alpha$ we have the following:

$$\left[\sum_i x_i^2, x \right] = x(2\nu(\alpha) + \langle \alpha, \alpha \rangle)$$

which we prove as:

$$\begin{aligned}\left[\sum_i x_i^2, x \right] &= \sum_i \alpha(x_i) x x_i + x_i \alpha(x_i) x \\ &= 2\alpha(x_i) x x_i + \alpha(x_i)^2 x \\ &= x(2\nu(\alpha) + \langle \alpha, \alpha \rangle)\end{aligned}$$

□

5.2. **Casimir element of $\widetilde{\mathfrak{g}}$.** In our case, the Kac–Moody algebra is $\widetilde{\mathfrak{g}}$. We have two expressions for $\widehat{C} \in U\widetilde{\mathfrak{g}}$ which follow from Corollary 5.1.

$$\widehat{C} = 2m(c + h^\vee)d + C + 2 \left(\sum_{\substack{\alpha \in R_+ \\ n > 0}} e_{-\alpha}(-n) e_\alpha(n) + e_\alpha(-n) e_{-\alpha}(n) + \sum_{\substack{n > 0 \\ i}} x_i(-n) x_i(n) \right) \tag{5.1}$$

where $\{x_i\}$ is an orthonormal basis of $\mathfrak{h}$, R is the root system of $(\mathfrak{g}, \mathfrak{h})$ (see §1.1), $e_{-\alpha} = d_\alpha f_\alpha$ is dual vector to e_α and C is the Casimir element for $\mathfrak{g}$.

Second expression for $\widehat{C}$ is more useful for future applications. Let B be an orthonormal basis of $\mathfrak{g}$. Then

$$\begin{aligned}\widehat{C} &= 2m(c+h^\vee)d + \sum_{a\in B}\left(\sum_{n>0} 2a(-n)a(n) + a(0)^2\right) \\ &= C + 2m(c+h^\vee)d + 2\sum_{\substack{a\in B\\ n>0}} a(-n)a(n)\end{aligned} \tag{5.2}$$

5.3. Let M be a highest weight $\widetilde{\mathfrak{g}}$–module of highest weight $\widetilde{\lambda} = \lambda + k\Lambda_0 + l\delta$. Let $m \in M$ be the highest weight vector of M. Then for any $u \in U\widetilde{\mathfrak{g}}$, we have:

$$d(u.m) = l(u.m) + [d,u].m$$
$$L_0(u.m) = -\Delta_k(\lambda)(u.m) + [L_0,u].m$$

where L_0 is defined by:

$$L_0 := -\frac{1}{2m(c+h^\vee)}\left(C + 2\sum_{\substack{a\in B\\ n>0}} a(-n)a(n)\right) \tag{5.3}$$

and $\Delta_k(\lambda)$ is given by (-1) the action of L_0 on m:

$$\Delta_k(\lambda) = \frac{\langle \lambda+2\rho, \lambda\rangle}{2m(k+h^\vee)} \tag{5.4}$$

Definition. The zeroth Sugawara operator is an element of a certain completion of $U\widehat{\mathfrak{g}}$ given by (5.3).

Theorem. *Let M be a highest weight module over $\widetilde{\mathfrak{g}}$ of highest weight $\widetilde{\lambda} = \lambda + k\Lambda_0 + l\delta$. Then we have*

$$d = L_0 + \Delta_k(\lambda) + l \tag{5.5}$$

where L_0 is given by (5.3) *and $\Delta_k(\lambda)$ is given by* (5.4).

5.4. **Operator KZ.** From this section onwards, we will only consider the evaluation at 1. To be more precise V is a finite–dimensional $\mathfrak{g}$–module and $V(z) = V[z,z^{-1}]$ is $\widetilde{\mathfrak{g}}$–module defined by:

$$x(r) \mapsto z^r x \qquad\qquad d \mapsto z\frac{d}{dz} \qquad\qquad c \mapsto 0$$

Let us fix $\widetilde{\lambda} = \lambda + k\Lambda_0 + l\delta \in \widetilde{\mathfrak{h}}^*$ and $\mu \in \mathfrak{h}^*$. In this section we derive a differential equation satisfied by the intertwiner $\Phi(z) : M_{\widetilde{\lambda}} \to M_{\widetilde{\lambda-\mu}}\widehat{\otimes}V(z)$. For $\xi \in V^*$ let us consider the following linear map:

$$\Psi^\xi(z) := (1\otimes\xi)\circ\Phi(z) : M_{\widetilde{\lambda}} \to M_{\widetilde{\lambda-\mu}}((z))$$

Definition. For $a \in \mathfrak{g}$ we define

$$a^+(z) = \sum_{n\geq 1} a(-n)z^{n-1}$$
$$a^-(z) = -\sum_{n\geq 0} a(n)z^{-n-1}$$

Then we have the following theorem.

Theorem. *The linear map Ψ^ξ satisfies the following differential equation:*

$$m(k+h^\vee)z\frac{\partial}{\partial z}\Psi^\xi(z) = z\left(\sum_{a\in B} a^+(z)\Psi^{a\xi} - \Psi^{a\xi}a^-(z)\right) + \frac{1}{2}\Psi^{C\xi} + c_{\lambda,\mu}\Psi^\xi$$

where C is the Casimir element of $\mathfrak{g}$ and $c_{\lambda,\mu}$ is a scalar given by:

$$c_{\lambda,\mu} := \frac{1}{2}\left(\langle\lambda+2\rho,\lambda\rangle - \langle\lambda-\mu+2\rho,\lambda-\mu\rangle\right)$$

Proof. We begin by recording the commutation relations between $x(n)$ and Ψ^ξ for any $x\in\mathfrak{g}$. Since Φ is an intertwiner we have:

$$\Phi\circ x(n) = (x(n)\otimes 1 + z^n 1\otimes x)\Phi$$

which composed with $(1\otimes\xi)$ gives:

$$[x(n),\Psi^\xi(z)] = z^n\Psi^{x\xi} \tag{5.6}$$

Similarly using $\Phi\circ d = (d\otimes 1 + 1\otimes d)\Phi$ we obtain:

$$z\frac{d}{dz}\Psi^\xi(z) = [\Psi^\xi(z), d]$$

By Theorem 5.3 we get:

$$z\frac{d}{dz}\Psi^\xi(z) = [\Psi^\xi(z), L_0] + \left(\Delta_k(\lambda) - \Delta_k(\lambda-\mu)\right)\Psi^\xi(z)$$

Using the definition of L_0 (5.3) and the commutation relations (5.6) we get:

$$\begin{aligned}
[\Psi^\xi(z), L_0] = \frac{1}{2m(k+h^\vee)}&\left[\sum_{a\in B}[a(0),\Psi^\xi(z)]a(0) + a(0)[a(0),\Psi^\xi(z)] + \right.\\
&\left. 2\left(\sum_{n>0}[a(-n),\Psi^\xi(z)]a(n) + a(-n)[a(n),\Psi^\xi(z)]\right)\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2m(k+h^\vee)}\left[\sum_{a\in B}\Psi^{a\xi}(z)a(0) + a(0)\Psi^{a\xi}(z) + 2\left(\sum_{n>0} z^{-n}\Psi^{a\xi}(z)a(n) + z^n a(-n)\Psi^{a\xi}(z)\right)\right]\\
&= \frac{1}{2m(k+h^\vee)}\left[\sum_{a\in B} a(0)\Psi^{a\xi}(z) - \Psi^{a\xi}(z)a(0) + 2za^+(z)\Psi^{a\xi}(z) - 2z\Psi^{a\xi}(z)a^-(z)\right]\\
&= \frac{1}{2m(k+h^\vee)}\left[2z\left(\sum_{a\in B} a^+(z)\Psi^{a\xi}(z) - \Psi^{a\xi}(z)a^-(z)\right) + \Psi^{C\xi}(z)\right]
\end{aligned}$$

since $C=\sum a^2$. This computation directly implies the statement of the theorem and we are done. $\square$

5.5. **Trigonometric KZ.** Recall the multicomponent fusion operator $\mathcal{J}^{1,\cdots,N}(\widetilde{\lambda}, z)$ is defined as (see Definition 2.6):

$$\mathcal{J}(\widetilde{\lambda}, z) = J\left(-\widetilde{\lambda} - \widehat{\rho} + \frac{1}{2}\sum_{j=1}^{N} h^{(j)}, z\right)$$

In order to spell out the definition of the multicomponent fusion operator, we fix some notations. Let $V_1, \cdots, V_N$ be finite–dimensional $\mathfrak{g}$–modules and choose $v_i \in V[\mu_i]$. Fix $\widetilde{\lambda} = \lambda + k\Lambda_0 + l\delta \in \widetilde{\mathfrak{h}}^*$ as before. Define:

$$\widetilde{\gamma}_i := -\widetilde{\lambda} - \widehat{\rho} + \frac{1}{2}\left(\mu_1 + \cdots + \mu_i - \mu_{i+1} - \cdots - \mu_N\right) = \gamma_i - (k + h^\vee)\Lambda_0 - l\delta$$

Then $\mathcal{J}(\widetilde{\lambda}, z)(v_1 \otimes \cdots \otimes v_N)$ is the expectation value of the composition:

$$\Phi(z_1, \cdots, z_N) = \Phi^{v_1}_{\widetilde{\gamma}_1} \circ \cdots \circ \Phi^{v_N}_{\widetilde{\gamma}_N}$$

Let us define the following operators on $V_1(z_1)\vec{\otimes}\cdots\vec{\otimes}V_N(z_N)$ (see Definition 4.2 for $\vec{\otimes}$)

$$\nabla_i := mkz_i\frac{\partial}{\partial z_i} + \sum_{j\neq i}\frac{r_{ij}z_j + r_{ji}z_i}{z_i - z_j} + \overline{\lambda}^{(i)} \tag{5.7}$$

$$\nabla^0_i := mkz_i\frac{\partial}{\partial z_i} - \frac{1}{2}\left(\sum_{j<i} x_l^{(j)}x_l^{(i)} - \sum_{j>i} x_l^{j)}x_l^{(i)}\right) + \overline{\lambda}^{(i)} \tag{5.8}$$

where $\{x_l\}$ is an orthonormal basis of $\mathfrak{h}$ and $A^{(i)} := 1^{\otimes i-1} \otimes A \otimes 1^{\otimes N-i}$. Recall the notations $\overline{\gamma} = \nu(\gamma) \in \mathfrak{h}$. The Drinfeld r–matrix is defined as:

$$r = \frac{1}{2}\sum x_l \otimes x_l + \sum_{\alpha>0} e_\alpha \otimes e_{-\alpha}$$

Theorem. *For each $i = 1, \cdots, N$ we have*

$$\nabla_i\mathcal{J}^{1,\cdots,N} = \mathcal{J}^{1,\cdots,N}\nabla^0_i \tag{5.9}$$

The proof of this theorem is given in §5.6 – §5.9. We begin by unfolding the definitions and introducing some notations for convenience.

Notations used in the proof: We fix $V_1, \cdots, V_N$ and $v_i \in V_i[\mu_i]$ throughout and use the notations at the beginning of this section. To simplify the expressions we will have $M_i = M_{\widetilde{\gamma}_i}$ and $\mathbf{1}_i = \mathbf{1}_{\widetilde{\gamma}_i}$. Let us write $\Phi_i(z_i) = \Phi^{v_i}_{\widetilde{\gamma}_i}(z_i)$. For a fixed linear from $\xi_i \in V_i^*$ we denote by $\Psi_i^{\xi_i}$ the composition $(1 \otimes \xi_i) \circ \Phi_i$ as in §5.4. We will drop the subscript $1, \cdots, N$ from $\mathcal{J}$ and for a linear form $\xi = \xi_1 \otimes \cdots \otimes \xi_N \in (V_1 \otimes \cdots \otimes V_N)^*$ we will write $\mathcal{J}^\xi$ for the composition $\xi \circ \mathcal{J}$:

$$\mathcal{J}^\xi(\underline{z}) := \left\langle \mathbf{1}_0^*, \Psi_1^{\xi_1} \cdots \Psi_N^{\xi_N}\mathbf{1}_N\right\rangle$$

With these conventions in mind, let us compute the right–hand side of (5.9).

$$\mathcal{J}(\widetilde{\lambda},\underline{z})\nabla_i^0(v_1\otimes\cdots\otimes v_N)=\left(\langle\lambda,\mu_i\rangle-\frac{1}{2}\left(\sum_{j<i}\langle\mu_j,\mu_i\rangle-\sum_{j>i}\langle\mu_j,\mu_i\rangle\right)\right)\mathcal{J}(\widetilde{\lambda},\underline{z})$$

Note that by linearity it suffices to prove (5.9) when evaluated on $v_1\otimes\cdots\otimes v_N$. Thus equation (5.9) is equivalent to the following:

$$(\nabla_i-p_i)\,\mathcal{J}(\widetilde{\lambda},\underline{z})(v_1\otimes\cdots\otimes v_N)=0$$

where p_i is the scalar obtained above:

$$p_i=\left(\langle\lambda,\mu_i\rangle-\frac{1}{2}\left(\sum_{j<i}\langle\mu_j,\mu_i\rangle-\sum_{j>i}\langle\mu_j,\mu_i\rangle\right)\right)$$

Evaluating this equation at a linear form ξ we obtain an equivalent version:

$$mkz_i\frac{\partial\mathcal{J}^\xi(\underline{z})}{\partial z_i}=-\xi\left(\left(\sum_{j\neq i}\frac{r_{ij}z_j+r_{ji}z_i}{z_i-z_j}+\lambda^{(i)}-p_i\right)\mathcal{J}(\underline{z})\right)\tag{5.10}$$

5.6. Let us begin by computing $\dfrac{\partial\mathcal{J}^\xi(\underline{z})}{\partial z_i}$ using Theorem 5.4.

$$\begin{aligned}z_i\frac{\partial\mathcal{J}^\xi(\underline{z})}{\partial z_i}&=\left\langle\mathbf{1}_0^*,\Psi^{\xi_1}(z_1)\cdots\Psi_{i-1}^{\xi_{i-1}}(z_{i-1})\left(z_i\frac{\partial\Psi_i^{\xi_i}(z_i)}{\partial z_i}\right)\Psi_{i+1}^{\xi_{i+1}}(z_{i+1})\cdots\Psi_N^{\xi_N}(z_N)\mathbf{1}_N\right\rangle\\&=\frac{-1}{mk}\left\langle\mathbf{1}_0^*,\cdots\left(z_i\left(\sum_{a\in B}a^+(z_i)\Psi^{a\xi_i}-\Psi_i^{a\xi_i}a^-(z_i)\right)+\frac{1}{2}\Psi^{C\xi_i}+c_i\Psi_i^{\xi_i}\right)\cdots\mathbf{1}_N\right\rangle\end{aligned}$$

where we have used the fact that $\widetilde{\gamma}_i(c)=-k-h^\vee$. The scalar c_i is given by (see Theorem 5.4)

$$c_i=\frac{1}{2}\left(\langle\gamma_i+2\rho,\gamma_i\rangle-\langle\gamma_{i-1}+2\rho,\gamma_{i-1}\rangle\right)$$

which one can easily calculate to be $-p_i$. We break the above expression into three terms:

$$T_1=\left\langle\mathbf{1}_0^*,\Psi^{\xi_1}(z_1)\cdots\Psi_{i-1}^{\xi_{i-1}}\left(z_i\sum_{a\in B}a^+(z_i)\Psi^{a\xi_i}(z_i)\right)\Psi_{i+1}^{\xi_{i+1}}(z_{i+1})\cdots\Psi_N^{\xi_N}(z_N)\mathbf{1}_N\right\rangle$$

$$T_2=\left\langle\mathbf{1}_0^*,\Psi^{\xi_1}(z_1)\cdots\Psi_{i-1}^{\xi_{i-1}}\left(z_i\sum_{a\in B}\Psi_i^{a\xi_i}(z_i)a^-(z_i)\right)\Psi_{i+1}^{\xi_{i+1}}(z_{i+1})\cdots\Psi_N^{\xi_N}(z_N)\mathbf{1}_N\right\rangle$$

$$T_3=\left\langle\mathbf{1}_0^*,\Psi^{\xi_1}(z_1)\cdots\Psi_{i-1}^{\xi_{i-1}}\left(\frac{1}{2}\Psi_i^{C\xi_i}(z_i)-p_i\Psi_i^{\xi_i}(z_i)\right)\Psi_{i+1}^{\xi_{i+1}}(z_{i+1})\cdots\Psi_N^{\xi_N}(z_N)\mathbf{1}_N\right\rangle$$

Note that T_3 is same as $\xi\left(\left(\frac{1}{2}C^{(i)}-p_i\right)\mathcal{J}\right)$. Thus we have to prove the following:

$$-mkz_i\frac{\partial\mathcal{J}^{\xi}(\underline{z})}{\partial z_i}=T_1-T_2+\xi\left(\left(\frac{1}{2}C^{(i)}-p_i\right)\mathcal{J}(\underline{z})\right) \tag{5.11}$$

5.7. Let us begin by computing T_1. In order to do so, we need the commutation relation between $a^{\pm}(z)$ and Ψ^{ξ}. These follow from $[x(n),\Psi^{\xi}]=z^n\Psi^{x\xi}$ given in (5.6):

$$[a^{\pm}(\zeta),\Psi^{\xi}(z)]=\frac{\Psi^{a\xi}(z)}{z-\zeta} \tag{5.12}$$

Hence we get (the summation over $a\in B$ is assumed):

$$T_1=z_i\left\langle\mathbf{1}_0^*,\Psi^{\xi_1}(z_1)\cdots\Psi_{i-1}^{\xi_{i-1}}(z_{i-1})a^+(z_i)\Psi^{a\xi_i}(z_i)\Psi_{i+1}^{\xi_{i+1}}(z_{i+1})\cdots\Psi_N^{\xi_N}(z_N)\mathbf{1}_N\right\rangle$$

$$=z_i\left\langle\mathbf{1}_0^*,a^+(z_i)\Psi^{\xi_1}(z_1)\cdots\Psi_{i-1}^{\xi_{i-1}}(z_{i-1})\Psi_i^{a\xi_i}\Psi_{i+1}^{\xi_{i+1}}(z_{i+1})\cdots\Psi_N^{\xi_N}(z_N)\mathbf{1}_N\right\rangle$$

$$-z_i\sum_{j=1}^{i-1}\left\langle\mathbf{1}_0^*,\cdots[a^+(z_i),\Psi_j^{\xi_j}(z_j)]\cdots\Psi_i^{a\xi_i}(z_i)\cdots\mathbf{1}_N\right\rangle$$

Recall from Definition 5.4 that $a^+(z)=\sum_{n\geq 1}a(-n)z^{n-1}$. Hence the first term above is zero and we get:

$$T_1=-z_i\sum_{\substack{1\leq j\leq i-1\\ a\in B}}\frac{1}{z_j-z_i}\left\langle\mathbf{1}_0^*,\cdots\Psi_j^{a\xi_j}(z_j)\cdots\Psi_i^{a\xi_i}(z_i)\cdots\mathbf{1}_N\right\rangle \tag{5.13}$$

5.8. It remains to compute T_2. Using the computation similar to the previous section, we have (again summation over $a\in B$ is assumed):

$$T_2=z_i\left\langle\mathbf{1}_0^*,\Psi^{\xi_1}(z_1)\cdots\Psi_{i-1}^{\xi_{i-1}}(z_{i-1})\Psi_i^{a\xi_i}\Psi_{i+1}^{\xi_{i+1}}(z_{i+1})\cdots\Psi_N^{\xi_N}(z_N)a^-(z_i)\mathbf{1}_N\right\rangle$$

$$+z_i\sum_{j=i+1}^{N}\left\langle\mathbf{1}_0^*,\cdots\Psi_i^{a\xi_i}(z_i)\cdots[a^-(z_i),\Psi_j^{\xi_j}(z_j)]\cdots\mathbf{1}_N\right\rangle$$

Using the fact that $a(n)\mathbf{1}_N=0$ for every $n\geq 1$ and the definition of $a^-(z)=-\sum_{n\geq 0}a(n)z^{-n-1}$ we get that the first term is same as $-T_2'$ where:

$$T_2'=\sum_{a\in B}\left\langle\mathbf{1}_0^*,\Psi^{\xi_1}(z_1)\cdots\Psi_{i-1}^{\xi_{i-1}}(z_{i-1})\Psi_i^{a\xi_i}(z_i)\Psi_{i+1}^{\xi_{i+1}}(z_{i+1})\cdots\Psi_N^{\xi_N}(z_N)a\mathbf{1}_N\right\rangle$$

while the last term is:

$$z_i\sum_{j=i+1}^{N}\frac{1}{z_j-z_i}\left\langle\mathbf{1}_0^*,\cdots\Psi_i^{a\xi_i}(z_i)\cdots\Psi_j^{a\xi_j}(z_j)\cdots\mathbf{1}_N\right\rangle$$

Hence we get

$$T_2=-T_2'+z_i\sum_{\substack{i+1\leq j\leq N\\ a\in B}}\frac{1}{z_j-z_i}\left\langle\mathbf{1}_0^*,\cdots\Psi_i^{a\xi_i}(z_i)\cdots\Psi_j^{a\xi_j}(z_j)\cdots\mathbf{1}_N\right\rangle \tag{5.14}$$

5.9. Finally in order to simplify the expression of T_2' we replace the tensor $\sum a\otimes a$ by $\sum x_l\otimes x_l + \sum_{\alpha>0} e_\alpha\otimes e_{-\alpha} + e_{-\alpha}\otimes e_\alpha$. Combined with the following facts:

(a) $x_l\mathbf{1}_N = \gamma_N(x_l)\mathbf{1}_N$.
(b) $\sum x_l\gamma_N(x_l) = \nu(\gamma_N)$.
(c) For each $\alpha>0$, $e_\alpha\mathbf{1}_N = 0$.

we obtain the following expression for T_2':

$$T_2' = -\left(\left(\overline{\gamma}_N^{(i)} + \sum_{j=1}^N e_\alpha^{(i)}e_{-\alpha}^{(j)}\right)\mathcal{J}\right)$$

Now we have:

$$\sum_{j=1}^N e_\alpha^{(i)}e_{-\alpha}^{(j)} = C^{(i)} + \overline{\rho}^{(i)} + \sum_{j\neq i} r_{ij} - \frac{1}{2}\sum_{\substack{l\\ j=1,\cdots,N}} x_l^{(i)}x_l^{(j)}$$

One can easily verify that $\sum_j x_l^{(i)}x_l^{(j)}$ acts by $(\overline{\gamma}_N - \overline{\gamma}_0)^{(i)}$. Combining these observations and the fact that $\gamma_0+\gamma_N = -2\lambda-2\rho$ we get:

$$T_2' = \xi\left(\overline{\lambda}^{(i)} - \sum_{j\neq i} r_{ij} - \frac{1}{2}C^{(i)}\right)\mathcal{J} \tag{5.15}$$

Substituting back the values of T_1 and T_2 from equations (5.13), (5.14) into (5.11) we obtain equation (5.10) and we are done with the proof of Theorem 5.5.

5.10. **Limit of equation** (5.9)**.** Let us consider the limiting case $z_1 >> z_2 >> \cdots >> z_N$ of the equation (5.9). Recall that the Drinfeld r–matrix for $\mathfrak{g}$ is defined by:

$$r = \frac{1}{2}\sum_l x_l\otimes x_l + \sum_{\alpha>0} e_\alpha\otimes e_{-\alpha}$$

If we let $z_k/z_l \to 0$ for every $k>l$, then we obtain:

$$\sum_{j\neq i}\frac{r_{ij}z_j + r_{ji}z_i}{z_i - z_j} \longrightarrow -\sum_{j<i} r_{ij} + \sum_{j>i} r_{ji}$$

Let us consider the case $N=2$. In this case the multicomponent fusion operator is same as the fusion operator:

$$\mathcal{J}^{1,2}(\widetilde{\lambda},\underline{z}) = J_{V_1,V_2}\left(-\widetilde{\lambda} - \widehat{\rho} + \frac{1}{2}(h^{(1)}+h^{(2)})\right) \in \mathrm{End}(V_1\otimes V_2)[[z_2/z_1]]$$

Let $\mathcal{J}(\widetilde{\lambda})$ be the constant term of $\mathcal{J}(\widetilde{\lambda},\underline{z})$. Equation (5.9) for $i=2$ then becomes:

$$\left(1\otimes\overline{\lambda} - \frac{1}{2}\sum x_l\otimes x_l - \sum_{\alpha>0} e_{-\alpha}\otimes e_\alpha\right)\mathcal{J}(\widetilde{\lambda}) = \mathcal{J}(\widetilde{\lambda})\left(1\otimes\overline{\lambda} - \frac{1}{2}\sum_l x_l\otimes x_l\right)$$

Since $\mathcal{J}$ is of weight zero, we evaluate both sides on the weight space $(V_1 \otimes V_2)[\mu_1+\mu_2]$ and use the following equality of operators on this weight space

$$\sum x_l \otimes x_l = (1 \otimes (\overline{\mu}_1 + \overline{\mu}_2) - 1 \otimes \sum x_l^2)$$

to obtain

$$[\mathcal{J}(\widetilde{\lambda}), 1 \otimes \widetilde{\theta}(\lambda)] = \left(\sum_{\alpha>0} e_{-\alpha} \otimes e_{\alpha} \right) \mathcal{J}(\widetilde{\lambda})$$

where

$$\widetilde{\theta}(\lambda) = -\overline{\lambda} + \frac{1}{2}(\overline{\mu}_1 + \overline{\mu}_2) - \frac{1}{2} \sum x_l^2 = \theta \left(-\lambda - \rho + \frac{1}{2}(\mu_1 + \mu_2) \right)$$

Recall the definition of $\theta(\lambda)$ from Theorem 1.8. This equation is precisely the ABRR equation (1.2). Since the solution to (1.2) is unique, and is given by the fusion operator for $\mathfrak{g}$ (see Theorem 1.8) we obtain:

Theorem. *The fusion operator $J(\widetilde{\lambda}, z)$ has the following form:*

$$J(\widetilde{\lambda}, z) = J(\lambda) + O(z)$$

In particular $J(\widetilde{\lambda}, z)$ is invertible.

5.11. **Application of Theorem 5.5 :I.** In this section we use Theorem 5.5 and Corollary 2.6 to prove the compatibility between the trigonometric KZ operators and dynamical Weyl group operators.

Theorem. *For each $i = 1, \cdots, N$ and $w \in B_{\mathrm{aff}}$ we have:*

$$\nabla_i(w\widetilde{\lambda}) \mathcal{A}_{w, V_1 \otimes \cdots \otimes V_N}(\widetilde{\lambda}) = \mathcal{A}_{w, V_1 \otimes \cdots \otimes V_N}(\widetilde{\lambda}) \nabla_i(\widetilde{\lambda})$$

(see Definition 2.6 for the notations).

Proof. We multiply both sides of the equation by $\mathcal{J}(\widetilde{\lambda})$ on the right and use Theorem 5.5, Corollary 2.6 to get:

$$\nabla_i(w\widetilde{\lambda}) \mathcal{A}_{w, V_1 \otimes \cdots \otimes V_N}(\widetilde{\lambda}) \mathcal{J}(\widetilde{\lambda}) = \mathcal{J}(w\widetilde{\lambda}) \nabla_i^0(w\widetilde{\lambda}) A_{w, V_1, \cdots, V_N}(\widetilde{\lambda})$$
$$\mathcal{A}_{w, V_1 \otimes \cdots \otimes V_N}(\widetilde{\lambda}) \nabla_i(\widetilde{\lambda}) \mathcal{J}(\widetilde{\lambda}) = \mathcal{J}(w\widetilde{\lambda}) \mathcal{A}_{w, V_1, \cdots, V_N}(\widetilde{\lambda}) \nabla_i^0(\widetilde{\lambda})$$

where $\mathcal{A}_{w, V_1, \cdots, V_N}$ is the multicomponent dynamical action (see Definition 2.6). The right–hand sides of both the equations above are in fact equal, which is easy to see since $\mathcal{A}_{w, V_1, \cdots, V_N}$ maps

$$\mathcal{A}_{w, V_1, \cdots, V_N}(\widetilde{\lambda}) : V_1[\mu_1] \otimes \cdots \otimes V_N[\mu_N] \to V_1[w\mu_1] \otimes \cdots \otimes V_N[w\mu_N]$$

and using the definition of ∇_i^0 given in (5.8). Finally since $\mathcal{J}$ is invertible (see Theorem 5.10) we obtain the desired assertion. □

5.12. **A generalization: factorizable systems.** Let $V_1, \cdots, V_N$ be $\mathfrak{g}$ modules and $r(z) \in \mathfrak{g} \otimes \mathfrak{g}$ be a meromorphic function satisfying the unitary condition $r(z) + r_{21}(-z) = 0$. Choose $s \in \mathfrak{g}$ such that $[s \otimes 1 + 1 \otimes s, r(z)] = 0$. Consider the following system of partial differential equations for a function $F(u_1, \cdots, u_N) \in V_1 \otimes \cdots \otimes V_N$:

$$\frac{\partial F}{\partial u_i} = \hbar \left(\sum_{j \neq i} r_{ij}(u_i - u_j) + s^{(i)} \right) F \tag{5.16}$$

Then we have the following theorem:

Theorem. *The system* (5.16) *is consistent if and only if*

$$[r_{12}(u_1 - u_2), r_{23}(u_2 - u_3)] + [r_{12}(u_1 - u_2), r_{13}(u_1 - u_3)] \\ + [r_{13}(u_1 - u_3), r_{23}(u_2 - u_3)] = 0 \tag{5.17}$$

The equation (5.17) is known as the *classical Yang–Baxter equation with spectral parameter.*

Proof. Again let $A_i = \sum_{j \neq i} r_{ij}(u_i - u_j) + s^{(i)}$. The system (5.16) is consistent if and only if we have, for every $i \neq j$:

$$\frac{\partial A_i}{\partial u_j} - \frac{\partial A_j}{\partial u_i} = [A_i, A_j]$$

Now we have the following computations:

(1) For $i \neq j$ the left–hand side is computed as:

$$\begin{aligned} \frac{\partial A_i}{\partial u_j} - \frac{\partial A_j}{\partial u_i} &= -r'_{ij}(u_i - u_j) + r'_{ji}(u_j - u_i) \\ &= 0 \end{aligned}$$

by the unitary condition $r_{ij}(u_i - u_j) + r_{ji}(u_j - u_i) = 0$.

(2) Next we compute the right–hand side :

$$[A_i, A_j] = \sum_{\substack{k \neq i \\ l \neq j}} [r_{ik}(u_i - u_k), r_{jl}(u_j - u_l)] + \sum_{k \neq i} [r_{ik}(u_i - u_k), s^{(j)}] + \sum_{l \neq j} [s^{(i)}, r_{jl}(u_j - u_l)]$$

Now it is easy to see that the last two terms vanish because of the constraint $[s \otimes 1 + 1 \otimes s, r(z)] = 0$ and the first term can be written as:

$$[A_i, A_j] = \sum_{k \neq i,j} [r_{ik}(u_i - u_k), r_{jk}(u_j - u_k)] + [r_{ij}(u_i - u_j), r_{jk}(u_j - u_k)] \\ + [r_{ik}(u_i - u_k), r_{ji}(u_j - u_i)]$$

□

Remark. In our case we have $r(u) = \dfrac{r_{21}e^u + r_{12}}{e^u - 1}$ and (5.9) is a factorizable system after making the change of variables $z_i = e^{u_i}$ in (5.16).

5.13. **Application of Theorem 5.5 : II.** Now we prove that the fusion operator defines an analytic function is a suitable domain. For this we rewrite equation (5.9) ignoring the scalar factor p_i, for a function $F(z_1, \cdots, z_N) \in V_1 \otimes \cdots \otimes V_N$ as:

$$mkz_i \frac{\partial F}{\partial z_i} = -\left(\sum_{j \neq i} \frac{r_{ij} z_j + r_{ji} z_i}{z_i - z_j} + \overline{\lambda}^{(i)} \right) F$$

Consider the change of variables $\zeta_i = z_{i+1}/z_i$ for $i = 1, \cdots, N-1$ and $\zeta_N = z_N$. Then we have:

$$mk\zeta_l \frac{\partial F}{\partial \zeta_l} = \left(\sum_{j=1}^{l} A_j \right) F \text{ for } l \neq N$$

where $A_i = \sum_{j \neq i} \dfrac{r_{ij} z_j + r_{ji} z_i}{z_i - z_j} + \overline{\lambda}^{(i)}$. For $l = N$ we have:

$$mk\zeta_N \frac{\partial F}{\partial \zeta_N} = -\left(\sum_{j=1}^{N} A_j \right) F = -\overline{\lambda} F$$

We replace F by $\zeta_N^{\lambda/mk} F$ to get $\dfrac{\partial F}{\partial \zeta_N} = 0$. Therefore the function F does not depend on ζ_N. Moreover we have:

$$\begin{aligned} A_i &= \sum_{j \neq i} \frac{r_{ij} z_j + r_{ji} z_i}{z_i - z_j} + \overline{\lambda}^{(i)} \\ &= \sum_{j<i} \frac{r_{ij} + r_{ji} \zeta_{i-1} \cdots \zeta_j}{\zeta_{i-1} \cdots \zeta_j - 1} + \sum_{j>i} \frac{r_{ij} \zeta_{j-1} \cdots \zeta_i + r_{ji}}{1 - \zeta_{j-1} \cdots \zeta_i} + \overline{\lambda}^{(i)} \end{aligned}$$

Thus we obtain the following equivalent system:

$$mk\zeta_l \frac{\partial F}{\partial \zeta_l} = \widetilde{A}_l(\zeta_1, \cdots, \zeta_N) F \tag{5.18}$$

where each $\widetilde{A}_l$ is regular function in the domain:

$$D := \{(\zeta_1, \cdots, \zeta_{N-1}) \in \mathbb{C}^{N-1} : |\zeta_i| < 1\}$$

The system (5.18) is a consistent system with normal crossing divisors at 0. Thus we can use the following theorem, which is well known in the theory of differential equations (see §5.14 for a proof):

Theorem. *The system* (5.18) *has a fundamental solution*

$$F = \zeta_1^{\widetilde{A}_1(0)} \cdots \zeta_{N-1}^{\widetilde{A}_{N-1}(0)} F_0(\zeta_1, \cdots, \zeta_{N-1})$$

where F_0 is computed as a formal power series and defines an analytic function in the domain D. In particular if $f(\underline{\zeta})$ is a vector valued solution of (5.18) *of the form:*

$$f(\underline{\zeta}) = \zeta_1^{\delta_1} \cdots \zeta_{N-1}^{\delta_{N-1}} f_0(\underline{\zeta})$$

where f_0 is a formal power series, then $f = F.v$ for a constant vector $v \in V_1 \otimes \cdots \otimes V_N$ and hence f_0 converges in D.

As a consequence of this theorem (and using Theorem 5.5) we obtain the following important

Corollary. *The fusion operator* $\mathcal{J}(\widetilde{\lambda}, \underline{z})$ *defines an analytic function in the domain* $|z_1| > \cdots > |z_N|$.

5.14. **Proof of Theorem 5.13.** We will give a proof of the Theorem 5.13 for the case $N = 1$, which can be easily extended to arbitrary N.

Let us consider the problem of solving a system

$$\frac{dF}{dz} = \frac{A(z)}{z}F$$

where $A(z) = \sum_{r\geq 0} A_r z^r$ defines an analytic function in the disc $D_a := \{z \in \mathbb{C} : |z| < a\}$ with values in $\mathcal{M}_n(\mathbb{C})$. We begin by writing a fundamental solution of the form:

$$\Phi(z) = H(z)z^{A_0}$$

with $H(z) = 1 + \sum_{r\geq 1} H_r z^r$. Assuming that the operator $m - \mathrm{ad}(A_0)$ is invertible on $\mathcal{M}_n(\mathbb{C})$, we can write a recursive system:

$$H_m = (m - \mathrm{ad}(A_0))^{-1} \sum_{r=1}^{m} A_r H_{m-r}$$

which solves the differential equation formally. We claim that the formal solution converges in the disc D_a. The proof is given in the following steps:

(a) Define a scalar valued function $\phi(z) = \sum_{i=1}^{\infty} ||A_i|| z^i$. A standard argument proves that $\phi(z)$ is convergent in $|z| < a$.

(b) Choose a positive constant c such that $||(m - ad(A_0))^{-1}|| < c$ for every $m \geq 1$. Define

$$y(z) := \frac{1}{1 - c\phi(z)}$$

The fact that $\phi(0) = 0$ implies that there is (possibly smaller) neighborhood of 0 on which $y(z)$ is convergent, say $|z| < a_1 \leq a$.

(c) Let $y(z) = \sum_{i=0}^{\infty} y_i z^i$ be power series expansion of $y(z)$ in the disc $|z| < a_1$. Then we have:

$$y_0 = 1 \text{ and } y_m = c\sum_{i=1}^{m} y_{m-i}||A_i||$$

(d) It follows from the definitions that $||H_m|| < y_m$ for every m. Hence convergence of $y(z)$ implies convergence of $H(z)$ in $|z| < a_1$.

(e) To complete the argument, observe that $H(z)$ is solution of following differential equation:

$$H(z)' = \frac{\sum_{i=0}^{\infty} B_i z^i}{z} H(z)$$

where each B_i is linear operator on $\mathcal{M}_n(\mathbb{C})$ given by: $B_0 = ad(A_0)$ and B_m is multiplication by A_m. Since this differential equation is defined in the disc $|z| < a$, the solution $H(z)$ has no singularities in $a_1 \leq |z| < a$.

6. Trigonometric q–KZ equations

In this section we obtain a difference equation satisfied by the multicomponent fusion operator for quantum affine algebras.[6]

6.1. The Drinfeld element. Let (H, R) be a quasi–triangular Hopf algebra. That is, the following axioms are satisfied:

$$\Delta^{21}(a) = R\Delta(a)R^{-1} \text{ for every } a \in H$$
$$\Delta \otimes 1(R) = R_{13}R_{23}$$
$$1 \otimes \Delta(R) = R_{13}R_{12}$$

Proposition. *Assuming (H, R) is a quasi–triangular Hopf algebra, we have:*

(1) $(\varepsilon \otimes 1)(R) = 1 \otimes 1 = (1 \otimes \varepsilon)(R)$
(2) $(S \otimes 1)(R) = R^{-1} = (1 \otimes S^{-1})(R)$

Proof. For (1) apply $(\varepsilon \otimes 1 \otimes 1)$ to $\Delta \otimes 1(R) = R_{13}R_{23}$ and use the counit axiom. For (2) apply $\mu_{12} \circ (S \otimes 1 \otimes 1)$ to the same hexagon axiom. In order to prove $S \otimes S(R) = R$, we apply $\mu_{23} \circ (S \otimes S \otimes 1)$ to $1 \otimes \Delta(R) = R_{13}R_{12}$. □

Theorem. *Let $u := m(S \otimes 1)(R_{21})$. Then we have:*

(1) *u is invertible with $u^{-1} = m(S^{-1} \otimes S)(R_{21})$.*
(2) *$S^2(x) = uxu^{-1}$ for every $x \in H$.*

Proof. We begin by proving that $ux = S^2(x)u$ for every $x \in H$. Let us write

$$\Delta^{(3)}(x) = f_k \otimes g_k \otimes h_k$$

Using $R \otimes 1(\Delta \otimes 1) \circ \Delta(x) = (\Delta^{21} \otimes 1) \circ \Delta(x)R \otimes 1$ we get (writing $R = a_i \otimes b_i$):

$$a_i f_k \otimes b_i g_k \otimes h_k = g_k a_i \otimes f_k a_i \otimes h_k$$

Now apply $\mu^{(3)} \circ (13) \circ (1 \otimes S \otimes S^2)$ to both sides of this equation to get:

$$S^2(h_k)S(g_k)S(b_i)a_i f_k = S^2(h_k)S(b_i)S(f_k)g_k a_i$$

Using $S(f_k)g_k \otimes h_k = 1 \otimes x$ we get that the right–hand side is same as $S^2(x)u$. Similarly using $f_k \otimes g_k S(h_k) = x \otimes 1$ we get that the left–hand side is same as ux and we are done.

To prove that $v = S^{-1}(b_i)S(a_i)$ is the inverse of u we write:

$$uv = uS^{-1}(b_j)S(a_j) = S(b_j)uS(a_j) = S(b_i b_j)a_i S(a_j) = 1$$

where we have used the fact that $S \otimes 1(R) = R^{-1}$. Finally $S^2(v)u = uvu^{-1}u = 1$ implies that u admits both left and right inverses, which thus have to be the same as v. □

[6] Notes based on the talk by Sachin Gautam

6.2. **Quantum Casimir operator.** Let $\mathfrak{g}$ be a Kac–Moody algebra and let $U_q\mathfrak{g}$ be the corresponding Drinfeld–Jimbo quantum group, defined as:

Definition. $U_q\mathfrak{g}$ is a unital associative algebra over $\mathbb{C}$ generated by q^h, e_i, f_i where $h \in \mathfrak{h}$ and $i \in I$. These generators are subjected to the following relations:

(QG1) $q^h q^{h'} = q^{h+h'}$ for every $h, h' \in \mathfrak{h}$.

(QG2) For each $i \in I$ and $h \in \mathfrak{h}$ we have:

$$q^h e_i q^{-h} = q^{\alpha_i(h)} e_i \qquad\qquad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$$

(QG3) $[e_i, f_j] = \delta_{ij} \dfrac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}}$

(QG4) For $i \neq j$ let $t = 1 - a_{ij}$. Then we have:

$$\sum_{s=0}^{t} (-1)^s \begin{bmatrix} t \\ s \end{bmatrix}_{q_i} e_i^{t-s} e_j e_i^s = 0$$

and similarly for $f's$.

The algebra $U_q\mathfrak{g}$ has a structure of a Hopf algebra with the comultiplication given by:

$$\Delta(e_i) = e_i \otimes k_i + 1 \otimes e_i$$
$$\Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i$$
$$\Delta(q^h) = q^h \otimes q^h$$

where $k_i := q_i^{h_i}$. The counit ε is given by:

$$\varepsilon(e_i) = \varepsilon(f_i) = 0 \qquad \varepsilon(q^h) = 1$$

This determines the antipode S:

$$S(e_i) = -e_i k_i^{-1} \qquad\qquad S(f_i) = -k_i f_i$$
$$S(q^h) = q^{-h}$$

The following proposition is fundamental in constructing an R–matrix for $U_q\mathfrak{g}$. The proof of this proposition is highly computational and is skipped in these notes. An interested reader can consult [6] for details.

Proposition. *Let $U_q\mathfrak{b}_\pm$ be the subalgebras of $U_q\mathfrak{g}$ generated by q^h and $\{e_i : i \in I\}$ (respectively $\{f_i : i \in I\}$). Then there exists a unique non–degenerate pairing $\langle ., . \rangle : U_q\mathfrak{b}_+ \times U_q\mathfrak{b}_- \to \mathbb{C}$ determined by:*

(1) $\left\langle q^h, q^{h'} \right\rangle = q^{\langle h, h' \rangle}$
(2) $\langle e_i, f_j \rangle = \delta_{ij} \frac{1}{q_i - q_i^{-1}}$
(3) $\langle 1, \cdot \rangle = \langle \cdot, 1 \rangle = \varepsilon(\cdot)$
(4) $\langle aa', b \rangle = \langle a \otimes a', \Delta^{21}(b) \rangle$
(5) $\langle a, bb' \rangle = \langle \Delta(a), b \otimes b' \rangle$

Theorem. *Let $R \in U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ be the canonical element determined by Proposition 6.2. Then $(U_q\mathfrak{g}, R)$ is a quasi–triangular Hopf algebra.*

Proof. We prove this theorem assuming the existence and uniqueness of the Drinfeld pairing. We begin by proving the hexagon axioms:

(1) We pair both sides of the equation $\Delta \otimes 1(R) = R_{13}R_{23}$ with a dual vector $b_k \otimes b_l \otimes a_t$, which simplifies to $\langle \Delta(a_t), b_k \otimes b_l \rangle = \langle a_t, b_k b_l \rangle$. Similarly the other hexagon axiom is equivalent to $\langle aa', b \rangle = \left\langle a \otimes a', \Delta^{21}(b) \right\rangle$.

(2) In order to prove that R intertwines Δ and Δ^{21}, we need a multiplicative identity: for every $x \in U^+$ and $y \in U^-$ we have:

$$yx = \left\langle S^{-1}(x^{(1)}), y^{(1)} \right\rangle \left\langle x^{(3)}, y^{(3)} \right\rangle x^{(2)} y^{(2)}$$

where $a^{(1)} \otimes a^{(2)} \otimes a^{(3)} = \Delta^{(3)}(a)$. This identity is proved in the following two steps: first one proves that the equation for x, x' implies the same for xx' (and similarly for y). This reduces to checking the identity on the generators of $U_q\mathfrak{g}$, which can be done directly.

(3) Finally let $x \in U^+$ and consider the product $R\Delta(x)$:

$$\begin{aligned} R\Delta(x) &= a_i x^{(1)} \otimes b_i x^{(2)} \\ &= a_i x^{(1)} \otimes x^{(3)} b_i^{(2)} \left\langle S^{-1}(x^{(2)}, b_i^{(1)} \right\rangle \left\langle x^{(4)}, b_i^{(3)} \right\rangle \end{aligned}$$

Use the hexagon axiom to write:

$$a_i \otimes b_i^{(1)} \otimes b_i^{(2)} \otimes b_i^{(3)} = a_i a_j a_k \otimes b_k \otimes b_j \otimes b_i$$

which yields:

$$\begin{aligned} R\Delta(x) &= a_i a_j a_k x^{(1)} \otimes x^{(3)} b_j \left\langle S^{-1}(x^{(2)}), b_k \right\rangle \left\langle x^{(4)}, b_i \right\rangle \\ &= x^{(4)} a_j S^{-1}(x^{(2)}) x^{(1)} \otimes x^{(3)} b_j \\ &= x^{(2)} \otimes x^{(1)} R \end{aligned}$$

as required. Note that we have used the fact that $\sum a_i \langle z, b_i \rangle = z$ for any $z \in U^+$ and the axioms of the Hopf algebra H.

□

As a consequence of the theorem above, we obtain the following important

Corollary. *Let u be the Drinfeld element of $U_q\mathfrak{g}$. Define $C^q := q^{2\rho} u^{-1}$. Then C^q is a central element. Moreover, V is a highest weight module over $U_q\mathfrak{g}$ of highest weight λ then*

$$C^q = q^{\langle \lambda + 2\rho, \lambda \rangle} Id_V$$

where $\rho \in \mathfrak{h}^*$ is any element satisfying $\rho(h_i) = 1$ for each i. Let us denote $\Delta(\lambda) = \langle \lambda + 2\rho, \lambda \rangle$.

6.3. **Quantum Sugawara operator.** We return to our situation of the Kac–Moody algebra $\widetilde{\mathfrak{g}}$. We have the following expression for the R–matrix (see Theorem 6.2)

$$\widetilde{\mathcal{R}} = q^{m(c \otimes d + d \otimes c)} q^{\sum x_l \otimes x_l} \sum_t u_t \otimes u^t$$

where $\{u_t\}$ is a homogeneous basis of $U_q\widehat{\mathfrak{n}}_+$ and $\{u^t\}$ is the dual basis of $U_q\widehat{\mathfrak{n}}_-$ with respect to the Drinfeld pairing given in Proposition 6.2.

Using our choice of $\widehat{\rho} = \rho + h^\vee\Lambda_0$ we obtain the following expression for the q–Casimir element:

$$\begin{aligned} C^q &= q^{2\rho+2mh^\vee d}u^{-1} \\ &= q^{2\rho+2md(c+h^\vee)}q^{\sum x_l^2}\sum_t S^{-1}(u^t)S(u_t) \end{aligned} \tag{6.1}$$

Note that since C^q acts by $q^{\Delta(\widetilde{\lambda})}$ on a highest weight module V of highest weight $\widetilde{\lambda}$, we get:

$$u = q^{2\overline{\rho}+2mh^\vee d}q^{-\Delta(\widetilde{\lambda})} \tag{6.2}$$

Note that in our case we have:

$$\Delta(\widetilde{\lambda}) = \langle\lambda+2\rho,\lambda\rangle + 2ml(k+h^\vee)$$

Theorem. *The following two operators are equal on a highest weight module V of highest weight $\widetilde{\lambda}$:*

$$q^{2md(k+h^\vee)} = q^{-2\overline{\rho}+\Delta(\widetilde{\lambda})}\left(\mu\left((q^{2mkd}\otimes 1)((S\otimes 1)(\mathcal{R}_{21}))(q^{-2mkd}\otimes 1)\right)\right)$$

where $\mathcal{R} = q^{-m(c\otimes d+d\otimes c)}\widetilde{\mathcal{R}}$.

Proof. Let us write $e^t = q^m$ and $\mathcal{R} = \alpha_i\otimes\beta_i$. Then the expression of $Y = S\otimes 1(\widetilde{\mathcal{R}}_{21})$ is:

$$Y = \sum_{a,b\geq 0}\frac{(-t)^{a+b}}{a!b!}S(\beta_i)d^ac^b\otimes c^ad^b\alpha_i$$

$$u = \mu(Y) = S(\beta_i)q^{-2mcd}\alpha_i$$

which implies the required equation, when compared with the expression (6.2). □

6.4. **More on R–matrix.** Recall that the universal R–matrix $\widetilde{\mathcal{R}}$ of $U_q\widetilde{\mathfrak{g}}$ is constructed using the Proposition 6.2. We define:

$$\mathcal{R} := q^{-m(c\otimes d+d\otimes c)}\widetilde{\mathcal{R}}$$

Explicitly written, we have:

$$\mathcal{R} = q^{\sum x_l\otimes x_l}\sum y_t\otimes y^t$$

where $\{y_t\}$ is a basis of $U_q\mathfrak{n}_+$ and $\{y^t\}$ is the dual basis of $U_q\mathfrak{n}_-$ under the Drinfeld pairing (see Proposition 6.2). We think of $\mathcal{R}$ as the R–matrix of $U_q\widehat{\mathfrak{g}}$. Let us define an automorphism D_z of $U_q\widehat{\mathfrak{g}}\otimes\mathbb{C}[z,z^{-1}]$ by:

$$D_z(h) = h \qquad D_z(e_i) = z^{\delta_{i0}}e_i \qquad D_z(f_i) = z^{-\delta_{i0}}f_i$$

and set $\mathcal{R}(z) := (D_z \otimes 1)(\mathcal{R}) \in U_q\widehat{\mathfrak{g}} \otimes U_q\widehat{\mathfrak{g}}[[z]]$. Further define:

$$\Delta_z(a) := (D_z \otimes 1)(\Delta(a))$$
$$\Delta'_z(a) := (D_z \otimes 1)(\Delta^{21}(a))$$

Theorem. *The R–matrix satisfies the following:*

(1) *For each $a \in U_q\widehat{\mathfrak{g}}$:*

$$\mathcal{R}(z)\Delta'_z(a)\mathcal{R}(z)^{-1} = q^{-m(c\otimes d+d\otimes c)}\Delta'_z(a)q^{m(c\otimes d+d\otimes c)}$$

(2) *Hexagon axioms hold:*

$$(\Delta \otimes 1)(\mathcal{R}(z)) = q^{m(d\otimes c\otimes 1)}\mathcal{R}_{13}(z)q^{-m(d\otimes c\otimes 1)}\mathcal{R}_{23}(z) \tag{6.3}$$

$$(1 \otimes \Delta)(\mathcal{R}(z)) = q^{-m(d\otimes c\otimes 1)}\mathcal{R}_{13}(z)q^{m(d\otimes c\otimes 1)}\mathcal{R}_{12}(z) \tag{6.4}$$

Proof. (1) is clear from the definitions and the intertwining property of $\widetilde{\mathcal{R}}$. In order to prove (6.3) use the hexagon axiom for $\widetilde{\mathcal{R}}$ and use the definition of $\mathcal{R}$:

$$q^{m(c\otimes 1\otimes d+1\otimes c\otimes d+d\otimes 1\otimes c+1\otimes d\otimes c)}\Delta \otimes 1(\mathcal{R}) \\ = q^{m(c\otimes 1\otimes d+d\otimes 1\otimes c)}\mathcal{R}_{13}q^{m(1\otimes c\otimes d+1\otimes d\otimes c)}\mathcal{R}_{23}$$

which yields:

$$\Delta \otimes 1(\mathcal{R}) = q^{-m(1\otimes c\otimes d)}\mathcal{R}_{13}q^{m(1\otimes c\otimes d)}\mathcal{R}_{23}$$

Now apply $D_z \otimes D_z \otimes 1$ and use the fact that conjugation with $q^{-m(1\otimes c\otimes d)}$ is same as that with $q^{m(d\otimes c\otimes 1)}$ on $\mathcal{R}_{13}$ to get (6.3). The equation (6.4) is proved similarly. □

6.5. **Quantum currents.** Let us recall the conventions of the loop representations. Let V be a finite–dimensional module over $U_q\widehat{\mathfrak{g}}$. We define $V(z)$ to be the representation of $U_q\widetilde{\mathfrak{g}}$ (or by restriction the module over $U_q\widehat{\mathfrak{g}}$) defined as: $V(z) := V[z, z^{-1}]$ and:

$$x \mapsto D_z(x) \qquad\qquad d \mapsto z\frac{d}{dz}$$

Now let V be a finite–dimensional $U_q\widehat{\mathfrak{g}}$–module and define:

$$L_V^{\pm}(z) \in U_q\widehat{\mathfrak{g}} \otimes \mathrm{End}(V)[[z^{\pm 1}]]$$

by:

$$L_V^+(z) := (1 \otimes \pi_V)(\mathcal{R}_{21}(z)) = (1 \otimes \pi_{V(z)})(\mathcal{R}_{21}) \tag{6.5}$$

$$L_V^-(z) := (1 \otimes \pi_V)(\mathcal{R}^{-1}(z^{-1})) = (1 \otimes \pi_{V(z)})(\mathcal{R}^{-1}) \tag{6.6}$$

6.6. **Operator q–KZ equations.** In this paragraph let $\widetilde{\lambda} = \lambda + k\Lambda_0 + l\delta$ and let $\mu \in \mathfrak{h}^*$. Consider the following intertwiner:

$$\Phi(z) : M_{\widetilde{\lambda}} \to M_{\widetilde{\lambda}-\mu} \widehat{\otimes} V(z)$$

Define:

$$p := q^{-2m(k+h^\vee)} \qquad \mathcal{D} := p^d \qquad U := q^{2(cdm-\overline{\rho})}u$$

Note that by (6.2) we have the following:

$$\mathcal{D} = U^{-1} q^{-\Delta(\widetilde{\lambda})} \text{ on } M_{\widetilde{\lambda}}$$

This observation, together with the fact that $\Phi(z)$ is an intertwiner implies that:

$$\Phi(p^{-1}z) = q^{\Delta(\widetilde{\lambda})-\Delta(\widetilde{\lambda}-\mu)}(U^{-1}\otimes 1)\Phi(z)U \tag{6.7}$$

In order to carry out the computation for $U^{-1}\otimes 1 \circ \Phi(z) \circ U$, we need the following two definitions:

Definition. Let H be a Hopf algebra and V_1, V_2, V_3 be H–modules.

(1) Let $*$ denote a right action of $H^{\otimes 3}$ on $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2 \otimes V_3)$ given by:

$$(x \otimes y \otimes z) * \Psi := (S(y) \otimes S(z)) \circ \Psi \circ x$$

(2) Let $\bullet$ denote the action of $H^{\otimes 2}$ on the same space of linear maps (written on the right), given by:

$$\Psi \bullet (a \otimes b) = (1 \otimes b) \circ \Psi \circ a$$

Note that we have the following:

$$(\Psi \bullet (a \otimes b)) \bullet (a' \otimes b') = \Psi \bullet (aa' \otimes b'b)$$

Theorem. *The intertwining operator Φ satisfies the following difference equation:*

$$\Phi(pz) = q^{\Delta(\widetilde{\lambda}-\mu)-\Delta(\widetilde{\lambda})} L_V^+(q^{km}pz)(1 \otimes q^{2\overline{\rho}}) \left(\Phi(z) \bullet L_V^-(z)^{-1}\right) \tag{6.8}$$

Proof. In light of the equation (6.7) it remains to compute $U^{-1} \otimes \Phi(z)U$. We begin by writing $\Phi(z) \circ U$ using the intertwining property (in the following equations $\widetilde{\mathcal{R}} = a_i \otimes b_i$):

$$\begin{aligned}
\Phi U &= \Phi S(b_i) a_i q^{2(cdm-\rho)} \\
&= \left((S \otimes S)\Delta^{21}(b_i)\right) \Phi a_i q^{2(cdm-\rho)} \\
&= \left(\left((1 \otimes \Delta^{21})(\widetilde{\mathcal{R}})\right) * \Phi\right) q^{2(cdm-\rho)} \\
&= \left(\widetilde{\mathcal{R}}_{13} * \left(\widetilde{\mathcal{R}}_{12} * \Phi\right)\right) q^{2(cdm-\rho)}
\end{aligned}$$

Now we use the fact that $\widetilde{\mathcal{R}}_{12} * \Phi(z) = (S \otimes 1)(\widetilde{\mathcal{R}}_{21})(u \otimes 1)\Phi$ to continue the computation:

$$\begin{aligned}
(U^{-1} \otimes 1)\Phi U &= (q^{-2(cdm-\rho)} \otimes 1)\left((u^{-1} \otimes 1)(1 \otimes S(b_i))(S(b_j) \otimes a_j)(u \otimes 1)\Phi a_i\right) q^{2(cdm-\rho)} \\
&= (q^{-2(cdm-\rho)} \otimes 1)\left((1 \otimes S(b_i))\widetilde{\mathcal{R}}_{21}^{-1}\Phi a_i\right) q^{2(cdm-\rho)}
\end{aligned}$$

Next we use the following computation of $(1\otimes S)(\widetilde{\mathcal{R}})$:

$$(1\otimes S)(\widetilde{\mathcal{R}}) = (\mathrm{Ad}(q^{-2\rho})\otimes 1)\widetilde{\mathcal{R}}^{-1}(q^{-2mh^\vee})$$

Note that on a tensor product of highest weight module with finite–dimensional module $\widetilde{\mathcal{R}}$ is same as $(1\otimes q^{mkd})\mathcal{R}$ which allows us to further write:

$$(1\otimes S)(\widetilde{\mathcal{R}}) = (\mathrm{Ad}(q^{-2\rho})\otimes 1)\left(\mathcal{R}^{-1}(q^{-2mh^\vee})(1\otimes q^{-mkd})\right)$$

Using these computations we proceed with simplifying $(U^{-1}\otimes 1)\Phi U$:

$$\begin{aligned}(U^{-1}\otimes 1)\Phi(z)U = \Big[&\Big((q^{-2(kdm-\rho)}\otimes 1)\mathcal{R}_{21}^{-1}(1\otimes q^{-mkd})\Phi(z)\Big)\\ &\bullet\Big((q^{-2\rho}\otimes 1)\mathcal{R}^{-1}(q^{-2mh^\vee})(q^{2\rho}\otimes q^{-mkd})\Big)\Big]\, q^{2(kdm-\rho)}\end{aligned}$$

$$= \Big[\Big((q^{-2(kdm-\rho)}\otimes 1)\mathcal{R}_{21}^{-1}(1\otimes q^{-mkd})\Phi(z)\Big)\bullet\Big((q^{-2\rho}\otimes 1)\mathcal{R}^{-1}(q^{-2mh^\vee})(q^{2kdm}\otimes q^{-mkd})\Big)\Big]$$

$$= \Big[\Big((q^{-2(kdm-\rho)}\otimes q^{-kdm})\mathcal{R}_{21}^{-1}(1\otimes q^{-mkd})\Phi(z)\Big)\bullet\Big((q^{-2\rho}\otimes 1)\mathcal{R}^{-1}(q^{-2mh^\vee})(q^{2kdm}\otimes 1)\Big)\Big]$$

$$= \Big[\Big((q^{2\rho}\otimes 1)\mathcal{R}_{21}^{-1}(q^{km})(q^{-2kdm}\otimes q^{-2mkd})\Phi(z)\Big)\bullet\Big((q^{-2\rho}\otimes 1)\mathcal{R}^{-1}(q^{-2mh^\vee})(q^{2kdm}\otimes 1)\Big)\Big]$$

$$= \Big[\Big((q^{2\rho}\otimes 1)\mathcal{R}_{21}^{-1}(q^{km})\Phi(z)\Big)\bullet\Big((q^{-2kdm}\otimes 1)(q^{-2\rho}\otimes 1)\mathcal{R}^{-1}(q^{-2mh^\vee})(q^{2kdm}\otimes 1)\Big)\Big]$$

$$= \Big[\Big((q^{2\rho}\otimes 1)\mathcal{R}_{21}^{-1}(q^{km})\Phi(z)\Big)\bullet\big((q^{-2\rho}\otimes 1)\mathcal{R}^{-1}(p)\big)\Big]$$

We finally simplify it to:

$$(U^{-1}\otimes 1)\Phi(z)U = \Big((1\otimes q^{-2\rho})L_V^+(q^{km}z)^{-1}\Phi(z)\Big)\bullet L_V^-(p^{-1}z)$$

which together with (6.7) finishes the proof of this theorem. $\square$

6.7. **Trigonometric q–KZ equations.** Recall that we denote by $\mathcal{J}(\widetilde{\lambda}, z_1,\cdots,z_N)$ the unshifted dynamical fusion operator. Let us define the following operators:

Definition. For each $\widetilde{\lambda}\in\widetilde{\mathfrak{h}}^*$ and $i=1,\cdots,N$ define:

$$\nabla_i^q := \mathcal{R}_{i+1,i}\left(\frac{z_{i+1}}{z_i}\right)\cdots\mathcal{R}_{N,i}\left(\frac{z_N}{z_i}\right)(q^{2\lambda})_i T_{i,p}\mathcal{R}_{i,1}\left(\frac{z_i}{z_i}\right)^{-1}\cdots\mathcal{R}_{i,i-1}\left(\frac{z_i}{z_{i-1}}\right)^{-1}$$

Further define the "Cartan part" of $\nabla_{i,0}^q$ as:

$$\nabla_{i,0}^q := q_i^{2\lambda}q^{-\sum_{j<i}(x_l)_j(x_l)_i+\sum_{j>i}(x_l)_j(x_l)_i}T_{i,p}$$

where $p=q^{2km}$ and $T_{i,p}$ is the multiplicative shift of argument operators:

$$T_{i,p}z_j = p^{\delta_{ij}}z_j$$

Note that the shift involved in the definitions above is different from the one used in the previous section (which was again denoted by p for lack of a better notation).

Theorem.

$$\nabla_i^q\mathcal{J}(\widetilde{\lambda},\underline{z}) = \mathcal{J}(\widetilde{\lambda},\underline{z})\nabla_{i,0}^q \tag{6.9}$$

We again begin by introducing some convenient notations and unfolding the assertion of (6.9) (see §5.5). Let us fix $V_1, \cdots, V_N$ finite–dimensional $U_q\widehat{\mathfrak{g}}$–modules and $v_i \in V_i[\mu_i]$. Define:

$$\widetilde{\gamma}_i := -\widetilde{\lambda} - \widehat{\rho} + \frac{1}{2}(\mu_1 + \cdots + \mu_i - \mu_{i+1} - \cdots - \mu_N)$$

Let $\Phi_i(z_i) := \Phi_{\widetilde{\gamma}_i}^{v_i}(z_i)$ denote the intertwiner. Then, by definition, the operators $\mathcal{J}(\widetilde{\lambda}, \underline{z})$ evaluated on the vector $v_1 \otimes \cdots \otimes v_N$ is given by:

$$\mathcal{J}(\widetilde{\lambda}, \underline{z})(v_1 \otimes \cdots \otimes v_N) = \langle \mathbf{1}_0^*, \Phi_1(z_1) \cdots \Phi_N(z_N)\mathbf{1}_N \rangle$$

which we denote by $\Phi(z_1, \cdots, z_N) \in (V_1 \otimes \cdots \otimes V_N)[[z_2/z_1, \cdots, z_N/z_{N-1}]]$.

Evaluating both sides of (6.9) on $v_1 \otimes \cdots \otimes v_N$ one obtains the following equation:

$$\Phi(z_1, \cdots, z_{i-1}, pz_i, z_{i+1}, \cdots, z_N) = c\mathcal{R}_{i,i-1}\left(\frac{pz_i}{z_{i-1}}\right) \cdots \mathcal{R}_{i,1}\left(\frac{pz_i}{z_1}\right)\left(q^{-2\lambda}\right)_i$$
$$\mathcal{R}_{N,i}\left(\frac{z_N}{z_i}\right)^{-1} \cdots \mathcal{R}_{i+1,i}\left(\frac{z_{i+1}}{z_i}\right)^{-1} \Phi(z_1, \cdots, z_N) \quad (6.10)$$

where c is a constant given by:

$$c = q^{\langle 2\lambda, \mu_i \rangle - \sum_{j<i} \langle \mu_j, \mu_i \rangle + \sum_{j>i} \langle \mu_j, \mu_i \rangle} \quad (6.11)$$

The proof of (6.10) is given in §6.8 – 6.10.

6.8. The idea of the proof is to use the operator q–KZ equation (Theorem 6.6). Recall that by the definition of $\widetilde{\gamma}_i$ the relevant level here is $-k - h^\vee$, by which we should replace k of Theorem 6.6. We begin by writing the left–hand side of (6.10):

$$T_{i,p}\Phi(\underline{z}) = \langle \mathbf{1}_0^*, \Phi_1(z_1) \cdots \Phi_{i-1}(z_{i-1})\Phi_i(pz_i)\Phi_{i+1}(z_{i+1}) \cdots \Phi_N(z_N)\mathbf{1}_N \rangle$$

Using Theorem 6.6 we get:

$$\Phi_i(pz_i) = \left(L_{V_i}^+(q^{(k-h^\vee)m}z_i)(q^{2\rho})_i \left(\Phi_i(z_i) \bullet L_{V_i}^-(z_i)^{-1}\right)\right)$$

Thus we obtain

$$T_{i,p}\Phi(\underline{z}) = q^{\Delta(\widetilde{\gamma}_{i-1}) - \Delta(\widetilde{\gamma}_i)} \langle \mathbf{1}_0^*, \Phi_1(z_1) \cdots \Phi_{i-1}(z_{i-1})$$
$$\left(L_{V_i}^+(q^{(k-h^\vee)m}z_i)(q^{2\rho})_i \left(\Phi_i(z_i) \bullet L_{V_i}^-(z_i)^{-1}\right)\right) \Phi_{i+1}(z_{i+1}) \cdots \Phi_N(z_N)\mathbf{1}_N \Big\rangle \quad (6.12)$$

One can easily check that the constant term of the equation above is same as c given in (6.11).

In order to proceed, we need the commutation relations between the intertwiners and the quantum currents $L^\pm$.

6.9. The following lemma follows easily from the hexagon axioms (6.3) and (6.4).

Lemma. *Let $\Phi : M_{\widetilde{\lambda}} \to M_{\widetilde{\lambda-\mu}} \otimes V_1(z_1)$ be an intertwiner and V_2 be another finite–dimensional representation of $U_q\widehat{\mathfrak{g}}$. Then we have:*

$$\mathcal{R}_{V_2,V_1}\left(q^{-km}\frac{z_2}{z_1}\right) L^+_{V_2}(z_2)_{1,3}\,(\Phi(z_1)\otimes 1) = (\Phi(z_1)\otimes 1)\circ L^+_{V_2}(z_2)$$

as operators $M_{\widetilde{\lambda}} \otimes V_2(z_2) \to M_{\widetilde{\lambda-\mu}} \otimes V_1(z_1) \otimes V_2(z_2)$.

Using this lemma, one can reduce the equation (6.12) to obtain:

$$T_{i,p}\Phi(\underline{z}) = c\mathcal{R}_{i,i-1}\left(\frac{pz_i}{z_{i-1}}\right)\cdots\mathcal{R}_{i,1}\left(\frac{pz_i}{z_1}\right)\left(q^{\gamma_0+2\rho}\right)_i \langle \mathbf{1}_0^*, \Phi_1(z_1)\cdots\Phi_{i-1}(z_{i-1})$$
$$\left(\Phi_i(z_i)\bullet L^-_{V_i}(z_i)^{-1}\right)\Phi_{i+1}(z_{i+1})\cdots\Phi_N(z_N)\mathbf{1}_N\Big\rangle \quad (6.13)$$

6.10. Next we make use of the following commutation between the quantum current L^-_V and the intertwiners. Again the assertion of the following lemma follows directly from the hexagon axioms (see (6.3) and (6.4)).

Lemma. *Let Φ be an intertwiner as before (see statement of Lemma 6.9) and V_2 be a finite–dimensional representation of $U_q\widehat{\mathfrak{g}}$. Then we have:*

$$(1\otimes\beta_j\otimes 1)\mathcal{R}^{-1}_{3,2}(\Phi\otimes 1)\Delta(\alpha_j) = (1\otimes\beta_j\otimes 1)(\Phi\otimes 1)(\alpha_j\otimes 1)$$

where we write $\mathcal{R} = \alpha_j \otimes \beta_j$.

Using this lemma we can reduce equation (6.13) to the following

$$T_{i,p}\Phi(\underline{z}) = c\mathcal{R}_{i,i-1}\left(\frac{pz_i}{z_{i-1}}\right)\cdots\mathcal{R}_{i,1}\left(\frac{pz_i}{z_1}\right)\left(q^{-2\lambda}\right)_i\mathcal{R}_{N,i}\left(\frac{z_N}{z_i}\right)^{-1}\cdots\mathcal{R}_{i+1,i}\left(\frac{z_{i+1}}{z_i}\right)^{-1}\Phi(\underline{z})$$

which finishes the proof of Theorem 6.7.

6.11. **A generalization.** Similar to the generalization of the trigonometric KZ equation to arbitrary factorizable systems given in §5.12 we have the following result due to Frenkel and Reshetikhin.

Theorem. *Let $V_1, \cdots, V_N$ be N finite–dimensional vector spaces and let $\mathcal{R}_{ij}(u) \in \mathrm{End}(V_i\otimes V_j)$ be given operators which are depend meromorphically on the complex parameter u. Further assume that we have $d_i \in \mathrm{End}(V_i)$ such that*

$$\mathcal{R}_{ij}(u)d_id_j = d_id_j\mathcal{R}_{ij}(u)$$

To this data we associate the following system of difference equations with step κ:

$$F(u_1,\cdots,u_{i-1},u_i+\kappa,u_{i+1},\cdots,u_N) = A_iF(u_1,\cdots,u_N) \quad (6.14)$$

for a function $F(u_1,\cdots,u_N)\in V_1\otimes\cdots\otimes V_N$. Here the operators A_i are defined by:

$$A_i = \mathcal{R}_{i,i-1}(u_i-u_{i-1}+\kappa)\cdots\mathcal{R}_{i,1}(u_i-u_1+\kappa)d_i\mathcal{R}_{N,i}(u_N-u_i)^{-1}\cdots\mathcal{R}_{i+1,i}(u_{i+1}-u_i)^{-1}$$

Then the system (6.14) *is consistent if, and only if the quantum Yang–Baxter equations hold:*

$$\mathcal{R}_{i,i+1}(u_i - u_{i+1})\mathcal{R}_{i,i+2}(u_i - u_{i+2})\mathcal{R}_{i+1,i+2}(u_{i+1} - u_{i+2}) = \\ \mathcal{R}_{i+1,i+2}(u_{i+1} - u_{i+2})\mathcal{R}_{i,i+2}(u_i - u_{i+2})\mathcal{R}_{i,i+1}(u_i - u_{i+1})$$

Index of almost–fixed notations

- **Notations for simple Lie algebras**
  - $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$.
  - $\langle .,. \rangle$ is an invariant, symmetric, non–degenerate bilinear form on $\mathfrak{g}$.
  - $\mathfrak{h}$ is a fixed Cartan subalgebra of $\mathfrak{g}$.
  - $R \subset \mathfrak{h}^*$ is the root system of the pair $(\mathfrak{g}, \mathfrak{h})$.
  - $\Delta = \{\alpha_i : i \in I\} \subset R$ is a fixed base of R.
  - $\nu : \mathfrak{h}^* \to \mathfrak{h}$ is the isomorphism determined by $\langle .,. \rangle$ restricted to $\mathfrak{h}$.
  - For $\gamma \in \mathfrak{h}^*$, we define $\gamma^\vee := 2\dfrac{\nu(\gamma)}{\langle \gamma, \gamma \rangle} \in \mathfrak{h}$
  - $d_i := \dfrac{\langle \alpha_i, \alpha_i \rangle}{2}$.
  - $\Delta^\vee = \{h_i := \alpha_i^\vee : i \in I\}$.
  - W is the Weyl group. P, Q are the weight/root lattices respectively.
  - $\theta \in \mathfrak{h}^*$ is the longest root. We set $m = \dfrac{\langle \theta, \theta \rangle}{2}$.
  - $\rho \in \mathfrak{h}^*$ is defined by $\rho(h_i) = 1$ for each $i \in I$. It is same as the sum of the fundamental weights, which is also equal to half sum of the positive roots. We set $h^\vee := 1 + \rho(\theta^\vee)$.
  - Chevalley generators of $\mathfrak{g}$ are denoted by $\{h_i, e_i, f_i : i \in I\}$ where $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ are chosen so as to have $\langle e_i, f_i \rangle = d_i^{-1}$.
- **Notations for affine Lie algebras**
  - $\widehat{\mathfrak{g}}$ is the central extension of $\mathfrak{g}[z, z^{-1}]$ with the bracket given by:
  $$[x(k), y(l)] = [x, y](k + l) + k\delta_{k+l,0} \langle x, y \rangle mc$$
  - $\widetilde{\mathfrak{g}}$ is the semi–direct product of $\widehat{\mathfrak{g}}$ with the derivation d given by: $[d, c] = 0$ and $[d, x(n)] = nx(n)$.
  - $\widetilde{h} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d \subset \widetilde{\mathfrak{g}}$ is the Cartan subalgebra.
  - $\langle .,. \rangle$ is extended to $\widetilde{\mathfrak{g}}$ by the following formulae:
    (1) $\langle .,. \rangle$ on $\widetilde{\mathfrak{h}}$ is same as the bilinear form on $\mathfrak{h}$ and $(d, \mathfrak{h}) = (c, \mathfrak{h}) = (c, c) = (d, d) = 0$; $(c, d) = \dfrac{1}{m}$.
    (2) $\langle x(n), y(m) \rangle = \delta_{n+m,0} \langle x, y \rangle$.
  - $\widetilde{\nu} : \widetilde{\mathfrak{h}}^* \to \widetilde{\mathfrak{h}}$ is the isomorphism determined by $\langle .,. \rangle$. More explicitly $\widetilde{\nu}(\lambda) = \nu(\lambda)$ for $\lambda \in \mathfrak{h}$ and $\widetilde{\nu}(\Lambda_0) = md$, $\widetilde{\nu}(\delta) = mc$ where $\Lambda_0, \delta \in \widetilde{\mathfrak{h}}^*$ are dual to c, d respectively.
  - $\widehat{\rho}$ is chosen as $\rho + h^\vee\Lambda_0$. Hence $\widetilde{\nu}(\widehat{\rho}) = \nu(\rho) + mh^\vee d$.
  - $\widehat{R}$ is the affine root system of the pair $(\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{h}})$:
  $$\widehat{R} = \{\alpha + n\delta : \text{ either } \alpha \in R, n \in \mathbb{Z} \text{ or } \alpha = 0, n \in \mathbb{Z}^\times\}$$
  - $\widehat{\Delta} := \{\alpha_i : i \in I\} \cup \{\alpha_0 := \delta - \theta\}$
  - $\widehat{\Delta}^\vee = \{h_i : i \in I\} \cup \{h_0 = c - \theta^\vee\}$.
  - Define $f_0 := e_\theta(-1)$ and $e_0 := f_\theta(1)$. Then $\{f_i, h_i, e_i : i \in \{0\} \cup I\}$ are the Chevalley generators of $\widehat{\mathfrak{g}}$.

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