PRACTICE MID TERM 1
ABSTRACT ALGEBRA (5590H)

Problem 1. Let m, n be two positive integers. Determine the number of group homomorphisms from $\mathbb{Z}/m\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$.

Problem 2. Let $X \in \mathrm{GL}_2(\mathbb{R})$ be the following matrix: $X = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Prove that the centralizer $C_{\mathrm{GL}_2\mathbb{R}}(X)$ of X in $\mathrm{GL}_2(\mathbb{R})$ consists of diagonal matrices. Recall that the centralizer is defined as: $C_G(x) = \{g \in G : gx = xg\}$.

Problem 3. Let H be a finite abelian group (written additively) such that order of every element $h \in H$, $h \neq 0$, is 2 (i.e, $2h = 0$ for every $h \in H$). Prove that either $|H| = 2$, or $\sum_{h \in H} h = 0$.

Problem 4. Compute the number of conjugacy classes in the dihedral group D_{24} (symmetries of regular 12-gon).

Problem 5. Let G be a finite group acting on a finite set X, such that the action is *free*: that is, $\mathrm{Stab}_G(x) = \{e\}$ for every $x \in X$. Prove that $|G|$ divides $|X|$.

Problem 6. (a) Let G be a finite group such that $G/Z(G)$ is cyclic. Prove that G is abelian (hence $G = Z(G)$). (b) Use the previous exercise to show that if G is a non-abelian group of size pq, then $Z(G) = \{e\}$ (here p, q are two distinct primes).