PRACTICE PROBLEMS FOR MID TERM 2
ABSTRACT ALGEBRA (5590H)

Topics: Sylow theorems §4.5. Direct and semidirect products §5.1, 5.4 and 5.5. Finite abelian groups §5.2. Automorphisms of groups §4.4. Alternating group §3.5, 4.6.

Problem 1. Let G be a finite group with 231 elements. Prove that (a) there is a unique Sylow 11–subgroup of G and it is contained in $Z(G)$ (b) there is a unique Sylow 7–subgroup.

Problem 2. List all finite abelian groups of size 300. In each case, give the invariant factors of the group.

Problem 3. Let G be a finite group and $\psi : G \to G$ be an automorphism. Let $x \in G$ and $y = \psi(x)$. Show that ψ gives a bijection between conjugacy classes of G containing x and y.

Problem 3.∗ Take $G = S_5$ and let $\psi \in \mathrm{Aut}_{\mathrm{gp}}(S_5)$. Use the previous exercise to prove the following. (i) ψ gives a permutation of the set of 2–cycles in S_5. (ii) There exists $a \in \{1, 2, 3, 4, 5\}$ such that

$$\psi((1\ k)) = (a\ b_k),\ \ k = 2, 3, 4, 5.$$

(iii) Let $\sigma \in S_5$ be given by $\sigma(1) = a$ and $\sigma(k) = b_k$ $(k = 2, 3, 4, 5)$. Then $\psi(x) = \sigma x \sigma^{-1}$. (Hence, every automorphism of S_5 is inner).

Problem 4. Let $p \in \mathbb{Z}_{\geq 2}$ be a prime number, and let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Let G be the following group with p^3 elements:

$$G = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{F}_p \right\}.$$

Show that $G \cong N \rtimes K$, where

$$N = \left\{ \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : y, z \in \mathbb{F}_p \right\} \qquad \text{and} \qquad K = \left\{ \begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : x \in \mathbb{F}_p \right\}.$$

Problem 4.∗ Note that $N \cong \mathbb{F}_p \times \mathbb{F}_p$ and $K \cong \mathbb{F}_p$. Compute the group homomorphism $\alpha : \mathbb{F}_p \to \mathrm{Aut}_{\mathrm{gp}}(\mathbb{F}_p \times \mathbb{F}_p)$ so that $G \cong N \rtimes_\alpha K$.

Problem 5. Let $A_4 \triangleleft S_4$ be the alternating group. Show that A_4 does not have a subgroup of size 6. (This is the standard counterexample to the converse of Lagrange's theorem: $H < G$ implies $|H|$ divides $|G|$, but k divides $|G|$ does not imply that there is a subgroup of size k.)