

EULER AND THE PENTAGONAL NUMBER THEOREM

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ABSTRACT. In this paper we give the history of Leonhard Euler's work on the pentagonal number theorem, and his applications of the pentagonal number theorem to the divisor function, partition function and divergent series. We have attempted to give an exhaustive review of all of Euler's correspondence and publications about the pentagonal number theorem and his applications of it.

Comprehensus: In hoc dissertatione damus historiam operis Leonhardi Euleri super theorma numerorum pentagonalium, et eius usus theoremae numerorum pentagonalium ad functioni divisorii, functioni partitione et seriebus divergentibus. Conati sumus dedisse recensum plenum omnium commerciorum epistolicarum et editionum Euleri circa theorema numerorum pentagonalium et eius applicationum ipsius.

1. INTRODUCTION

The pentagonal numbers are those numbers of the form $\frac{n(3n-1)}{2}$ for n a positive integer. They represent the number of distinct points which may be arranged to form superimposed regular pentagons with the same number of equally spaced points on the sides of each respective pentagon. The generalized pentagonal numbers are those numbers of the form $\frac{n(3n\pm 1)}{2}$ for n non-negative, i.e. the pentagonal numbers for n an integer. However, in this paper unless we indicate otherwise, by pentagonal number we will mean a generalized pentagonal number.

In this paper we will consider Leonhard Euler's work on the pentagonal number theorem and his applications of it to recurrence relations for the divisor function and the partition function, and to divergent series. We have attempted to give an exhaustive summary of Euler's correspondence and works that discuss the pentagonal number theorem. The pentagonal number theorem is the formal identity:

$$(1) \quad \prod_{m=1}^{\infty} (1 - x^m) = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(3n-1)}{2}},$$

and it is called the pentagonal number theorem because the exponents in the formal power series on the right-hand side of the equation are the pentagonal numbers. Formally, the set of all power series over the integers form a commutative \mathbf{Z} -algebra closed under logarithmic differentiation and integration. Analytically, the series on the right-hand side of (1) converges absolutely for $|x| < 1$; in fact, for the function

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ϕ defined by $\phi(x) = \prod_{m=1}^{\infty} (1 - x^m)$ for $|x| < 1$, Konrad Knopp [Knopp, 1913] shows that ϕ does not have an analytic continuation beyond the unit circle.

We can observe in fact that the identity (1) follows directly from the Jacobi triple product identity $\prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m-1}z^2)(1 + q^{2m-1}z^{-2}) = \sum_{n=-\infty}^{\infty} q^{n^2} z^{2n}$, for $q = x^{\frac{3}{2}}$ and $z^2 = -x^{1/2}$. This gives $\prod_{m=1}^{\infty} (1 - x^{3m})(1 - x^{3m-1})(1 - x^{3m-2}) = \sum_{n=-\infty}^{\infty} (-1)^n x^{3n(n+1)/2}$, that is (1). We further discuss the Jacobi triple product identity in Section 7, and also q -series and theta functions. Fabian Franklin [Franklin, 1881] gives a combinatorial proof of the pentagonal number theorem by proving an equivalent result about partitions. George E. Andrews [Andrews, 1983] generalizes Euler's proof of the pentagonal number theorem to prove that $1 - \sum_{m=1}^{\infty} (1 - zq)(1 - zq^2) \cdots (1 - zq^{m-1})z^{m+1}q^m = 1 + \sum_{n=1}^{\infty} (-1)^n (z^{3n-1}q^{\frac{n(3n-1)}{2}} + z^{3n}q^{\frac{n(3n+1)}{2}})$; analytically, these series converge absolutely for $|q| < 1$ and $|z| < |q|^{-1}$.

Following Euler's notation, in this paper we express by $\int n$ the divisor function, the sum of all the divisors of n including itself, and by $n^{(m)}$ the partition function, the number of ways of writing n as a sum of positive integers less than or equal to m , disregarding order.

2. BACKGROUND

The first time the pentagonal number theorem is mentioned in Euler's correspondence is in a letter from Daniel I Bernoulli to Euler on January 28, 1741, Letter XX in the Daniel I Bernoulli-Euler correspondence in P.-H. Fuß's collection *Correspondance mathématique et physique de quelques célèbres géomètres du XVIIIème siècle* [Fuß, 1843], OO140 in the annotated index of Euler's correspondence in the *Opera omnia* [Juškevič et al., 1975]. In this letter Bernoulli discusses a number of problems that Euler apparently had posed to him in earlier letters (the letters from Euler in the Daniel Bernoulli I-Euler correspondence are not extant); in particular, he mentions the problem of finding all the partitions of an integer. Concerning the pentagonal number theorem, Bernoulli writes: "The other problem, to transform the expression $(1 - \frac{1}{n})(1 - \frac{1}{n^2})(1 - \frac{1}{n^3})$ into the series $1 - \frac{1}{n} - \frac{1}{n^2} + \frac{1}{n^5} + \frac{1}{n^7} - \frac{1}{n^{12}} - \frac{1}{n^{15}} +$ etc. follows easily by induction, if one multiplied many factors from the given expression. The remaining of the series, in which prime numbers are seen, I do not see. This can be shown in a most pleasant investigation, together with tranquil pastime and the endurance of pertinacious labor, all three of which I lack". Thus Euler probably had mentioned the problem of expanding this infinite product into an infinite series in his last letter, on September 15, 1740.

The first paper in which Euler mentions the pentagonal number theorem is his "Observationes analyticae variae de combinationibus", presented to the St. Petersburg Academy on April 6, 1741 and published in 1751 in the *Commentarii academiae scientiarum imperialis Petropolitanae* [Euler, 1751b], E158 in the Eneström index [Eneström, 1913]. In this paper Euler introduces the generating function

$$(2) \quad \prod_{x=1}^{\infty} \frac{1}{1 - n^x} = 1 + n + 2n^2 + 3n^3 + 5n^4 + 7n^5 + 11n^6 + 15n^7 + 22n^8 + \text{etc.}$$

for the (unrestricted) partition function, i.e. $n^{(\infty)}$. In §36 of this paper, Euler says, "Here at the end of this dissertation a noteworthy observation should be made,

which however I have not been able to demonstrate with geometric rigor. I have observed namely for this infinite product:

$$(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5) \text{ etc.},$$

if expanded by multiplication, to produce this series:

$$1 - n - n^2 + n^5 + n^7 - n^{12} - n^{15} + n^{22} + n^{26} - n^{35} - n^{40} + n^{51} + \text{etc.},$$

where each of these occurs as a power of n , of which the exponents are contained in the form $\frac{3xx \pm x}{2}$. And if x is an odd number, the powers of n , which are $n^{\frac{3xx \pm x}{2}}$, will have the coefficient -1 ; and if x is an even number, then the powers $n^{\frac{3xx \pm x}{2}}$ will have the coefficient $+1$ ". In §37, Euler notes as well that the product of the power series on the right-hand side of (2) and the above power series is unity, since they are the series expansions of reciprocal infinite products.

The next time the expansion of the infinite product $\prod_{m=1}^{\infty} (1-x^m)$ into a series comes up in Euler's correspondence is in a letter from Euler to Niklaus I Bernoulli on September 1, 1742, Letter 3 in the Euler-Niklaus I Bernoulli correspondence in the *Opera omnia* [Fellmann and Mikhajlov, 1998], OO236. However, Euler does discuss the problem of expressing the series $s - \frac{s^3}{6} + \frac{s^5}{120} - \text{etc.}$ as the infinite product $s(1 - \frac{ss}{\pi\pi})(1 - \frac{ss}{4\pi\pi})(1 - \frac{ss}{9\pi\pi})$ etc. before the pentagonal number theorem in this letter, and this is also discussed in the previous two letters in the Euler-Niklaus I Bernoulli correspondence. Euler also notes in this letter that the coefficients of the terms in the series

$$1 + 1n + 2n^2 + 3n^3 + 5n^4 + 7n^5 + 11n^6 + 15n^7 + 22n^8 + 30n^9 + 42n^{10} + 56n^{11} + \text{etc.}$$

give the number of different ways in which the exponent of the term can be made by addition, i.e. that it is the generating function for the (unrestricted) partition function. Euler then writes: "This series moreover arises from division, if unity were divided by $(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5)$ etc., which product if expanded gives this expression

$$1 - n - n^2 + n^5 + n^7 - n^{12} - n^{15} + n^{22} + n^{26} - n^{35} - \text{etc.}$$

where the precise way in which the exponents proceed I have not been able to penetrate, although by induction I have concluded for no other exponents to occur, unless they are contained in the formula $(3xx \pm x)/2$; and this is such that the powers of n have the $+$ sign if the exponents arise with an even number substituted for x ".

Niklaus I Bernoulli replies to Euler on October 24, 1742, Letter 4 in the Euler-Niklaus I Bernoulli correspondence in the *Opera omnia* [Fellmann and Mikhajlov, 1998], OO237. Bernoulli discusses the pentagonal number theorem after mentioning the generating function $\prod_{m=1}^{\infty} \frac{1}{1-x^m}$ for the partition function. Bernoulli here observes, "In the expansion of the series

$$n^0 - n^1 - n^2 + n^5 + n^7 - n^{12} - n^{15} + n^{22} + n^{26} - n^{35} - \text{etc.}$$

which you have found to be equal to the product $(1-n)(1-nn)(1-n^3)$ etc., the differences of the exponents progress as 1, 1, 3, 2, 5, 3, 7, 4, 9, 5, etc., which numbers taken alternately are from the series 1, 3, 5, 7, 9, etc. and from the series 1, 2, 3, 4, 5, etc., which properties will perhaps be able to be demonstrated from the nature of this thing not only through induction; but into this matter it is not now free to inquire".

Euler next writes to Niklaus I Bernoulli on November 10, 1742, Letter 5 in the Euler-Niklaus I Bernoulli correspondence in the *Opera omnia* [Fellmann and Mikhaĭlov, 1998], OO238, the last letter in their correspondence that deals with the pentagonal number theorem. Euler writes: “This expression

$$(1 - n)(1 - n^2)(1 - n^3)(1 - n^4) \text{ etc.}$$

by expansion shall give the series

$$1 - n - n^2 + n^5 + n^7 - \text{etc.}$$

in which no other exponents occur unless they are contained in

$$\frac{3xx \pm x}{2},$$

which I have for my part concluded with legitimate induction, even if I have not been able to find a demonstration in any manner, however, although I have not devoted enough time to this. I however have found this expression

$$(1 - n)(1 - n^2)(1 - n^3)(1 - n^4) \text{ etc.}$$

to be able indeed to be transformed into this series

$$1 - \frac{n}{1 - n} + \frac{n^3}{(1 - n)(1 - n^2)} - \frac{n^6}{(1 - n)(1 - n^2)(1 - n^3)} + \text{etc.}$$

of which the value is equal precisely to the sum of the series

$$1 - n^1 - n^2 + n^5 + n^7 - n^{12} - n^{15} + \text{etc.}”$$

In this letter Euler also gives more general results on the infinite products $(1 + mz)(1 + m^2z)(1 + mz^3)(1 + mz^4) \text{ etc.}$ and $\frac{1}{(1-mz)(1-m^2z)(1-m^3z)(1-m^4z) \text{ etc.}}$. Euler shows that

$$\begin{aligned} & (1 + mz)(1 + m^2z)(1 + m^3z)(1 + m^4z) \text{ etc.} \\ = 1 + & \frac{mz}{1 - m} + \frac{m^3z^2}{(1 - m)(1 - m^2)} + \frac{m^6z^3}{(1 - m)(1 - m^2)(1 - m^3)} + \text{etc.} \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{(1 - mz)(1 - m^2z)(1 - m^3z) \text{ etc.}} \\ = 1 + & \frac{mz}{1 - m} + \frac{m^2z^2}{(1 - m)(1 - m^2)} + \frac{m^3z^3}{(1 - m)(1 - m^2)(1 - m^3)} + \text{etc.} \end{aligned}$$

The next time Euler discusses the pentagonal number theorem in his correspondence is in a letter to Christian Goldbach on October 15, 1743, Letter 74 in the Euler-Goldbach correspondence [Juškevič and Winter, 1965], OO788. Euler states here, “If these factors $(1 - n)(1 - n^2)(1 - n^3)(1 - n^4)(1 - n^5) \text{ etc.}$ are multiplied out onto infinity, the following series

$$1 - n^1 - n^2 + n^5 + n^7 - n^{12} - n^{15} + n^{22} + n^{26} - n^{35} - n^{40} + n^{51} + n^{57} - \text{etc.}$$

is produced, from which it is easily shown by induction that all of the terms are contained in the form $n^{\frac{3xx \pm x}{2}}$, and that they have the prefixed sign + when x is an even number, and the sign - when x is odd. I have however not yet found a method by which I could prove the identity of these two expressions. The Hr. Prof. Niklaus Bernoulli has also been able to prove nothing beyond induction”.

Goldbach replies to Euler's problem in a letter dated December 1743, Letter 75 in the Euler-Goldbach collection [Juškevič and Winter, 1965], OO789. Goldbach does not give any explicit ideas for how to tackle proving the pentagonal number theorem, and instead poses a new related problem. He says, "About the series $(1-n)(1-nn)(1-n^3)$ etc., another problem has occurred to me: Given an infinite series of terms A , with an order given for the varying signs $+$ and $-$ of the progression, to find a series B of such a nature that in the product AB the signs $+$ and $-$ succeed in the same order that they succeeded in A . This problem can easily be solved in the case $A = (1-n)(1-nn)(1-n^3)$ etc., although as it has already been noted the signs $+$ and $-$ alternate in an unusual manner, for if I set $B = (1-n^{\frac{1}{2}})(1-n^{\frac{3}{2}})(1-n^{\frac{5}{2}})$ etc., then A multiplied by B becomes a new series, which contains the same variation of the signs".

Euler writes back to Goldbach about this on January 21, 1744, Letter 76 in the Euler-Goldbach correspondence [Juškevič and Winter, 1965], OO790, in which he says, "Reflection about the expression $(1-n)(1-n^2)(1-n^3)$ etc., in view of the factors $(1-n^{\frac{1}{2}})(1-n^{\frac{3}{2}})(1-n^{\frac{5}{2}})$, which, having been composed, if expanded, yields an equal alternation of the signs $+$ and $-$, could perhaps be advantageous in other research; alone in the series, which I derived from this, I have not been able to make any use of it".

On April 5, 1746, Euler writes another letter to Goldbach about the pentagonal number theorem, Letter 102 in the Euler-Goldbach correspondence [Juškevič and Winter, 1965], OO816. In this letter he considers several problems, in particular the expansion of the infinite products $(1+a)(1+a^2)(1+a^4)(1+a^8)$ etc. and $(1-a)(1-a^2)(1-a^4)(1-a^8)$ etc. into series. Euler states as a theorem that "If it were $s = (1-na)(1-n^2a)(1-n^3a)(1-n^4a)(1-n^5a)$ etc. onto infinity, it will be

$$s = 1 - \frac{na}{1-n} + \frac{n^3a^2}{(1-n)(1-n^2)} - \frac{n^6a^3}{(1-n)(1-n^2)(1-n^3)} + \frac{n^{10}a^4}{(1-n)(1-n^2)(1-n^3)(1-n^4)} - \text{etc.}$$

and

$$\frac{1}{s} = 1 + \frac{na}{1-n} + \frac{n^2a^2}{(1-n)(1-n^2)} + \frac{n^3a^3}{(1-n)(1-n^2)(1-n^3)} + \frac{n^4a^4}{(1-n)(1-n^2)(1-n^3)(1-n^4)} + \text{etc.}$$

I believe I have also written that if one multiplies this product onto infinity

$$(1-a)(1-a^2)(1-a^3)(1-a^4)(1-a^5) \text{ etc.}$$

this series is produced

$$1 - a - a^2 + a^5 + a^7 - a^{12} - a^{15} + a^{22} + a^{26} - a^{35} - a^{40} + \text{etc.},$$

where the order of the exponents is very peculiar, and also by induction it may be determined that all are contained in the form $\frac{3xx \pm x}{2}$, though I have not yet been able to expose the rule which has been observed from the nature of this matter".

Later, in a letter to Jean le Rond d'Alembert on December 30, 1747, Letter 11 in the Euler-d'Alembert correspondence in the *Opera omnia* [Juškevič and Taton, 1980], OO23, Euler writes that he has learned from de Maupertuis (the President of the Berlin Academy) that d'Alembert wants to leave his mathematical research for

some time to regain his health. Euler goes on to say, “If in your spare time you should wish to do some research which does not require much effort, I will take liberty to propose the expression $(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc., which upon expansion by multiplication gives the series

$$1 - x^1 - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + x^{57} - x^{70} - x^{77} + \text{etc.}$$

which would seem very remarkable to me because of the law which we easily discover within it, but I do not see how this law may be deduced without induction of the proposed expression”. As a postscript to the letter, Euler writes, “If we put $s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc., I can show that

$$s = 1 - \frac{x}{1-x} + \frac{x^3}{(1-x)(1-x^2)} - \frac{x^6}{(1-x)(1-x^2)(1-x^3)} + \frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)} - \text{etc.}”$$

d’Alembert replies to Euler in a letter on January 20, 1748, Letter 12 in the Euler-d’Alembert correspondence in the *Opera omnia* [Juškevič and Taton, 1980], OO24. d’Alembert says, “regarding the series of which you have spoken, it is very peculiar, and I have reflected on it for a while, but I only see induction to show it. At the end, no one is deeper and better versed on these matters than you”.

In §323 (in Chapter XVI, “De partitione numerorum”) of the *Introductio in analysin infinitorum* [Euler, 1748], E101, published in 1748, Euler notes in his discussion on the generating function $\prod_{m=1}^{\infty} \frac{1}{1-x^m}$ of the partition function that, “in particular it should be observed for there to be a ladder relationship in the denominator, for if indeed the factors of the denominator are multiplied successively into each other, it will advance

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \text{etc.},$$

in which series if it were considered attentively, for no other powers of x to be discovered, unless the exponents of them are contained in the formula $\frac{3nn \pm n}{2}$; and too, if n is an odd number, the powers will be negative; positive if indeed n is an even number”.

3. EULER’S PROOF OF THE PENTAGONAL NUMBER THEOREM

In a letter from Goldbach to Euler on April 15, 1747, Letter 114 in the Euler-Goldbach correspondence [Juškevič and Winter, 1965], OO828, Goldbach responds to Euler’s previous letter that had given a recurrence relation for the divisor function (cf. Section 4). In that letter, Euler had remarked that his proof assumes the pentagonal number theorem, which he had not been able to rigorously prove. Goldbach declares that, “The observation which you have communicated to me seems to me already through the given induction proved to the extent that one can believe in its truth one-hundred to one. Moreover, it has already been noted earlier that $A \dots (1-x)(1-xx)(1-x^3)(1-x^4)$ etc. = $B \dots 1-x-xx+x^5+x^7-x^{12}-x^{15}+$ etc. and I remember that from this I came to the simple conclusion that when the powers of x in B are doubled, and with

$$C = 1 - xx - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + \text{etc.}$$

then it must be

$$\frac{C}{B} = (1-x)(1-x^3)(1-x^5)(1-x^7) \text{ etc.}”$$

Euler replies to Goldbach on May 6, 1747, Letter 115 in the Euler-Goldbach correspondence [Juškevič and Winter, 1965], OO829, saying: “The observation which was made about the identity

$$\begin{aligned} A \dots (1-x)(1-x^2)(1-x^3)(1-x^4) \text{ etc.} = \\ B \dots 1-x-x^2+x^5+x^7-\text{etc.} \end{aligned}$$

that if

$$C = 1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + \text{etc.},$$

then $\frac{C}{B} = (1-x)(1-x^3)(1-x^5) \text{ etc.}$, I still remember well. I have however neither from this nor from other considerations been able to display the identity between the formulas A and B properly; for the fact that $A = B$ and that the exponents of x in B continue only according to the series, I have also only been able to conclude by induction, which I however have continued so far, that I consider the matter completely true; I would be very enthusiastic to see a direct proof of this matter, which would certainly lead to the discovery of many other beautiful properties of numbers; hitherto all of my pains have been for nothing”.

At last in a letter from Euler to Goldbach on June 9, 1750, Letter 144 in the Euler-Goldbach correspondence [Juškevič and Winter, 1965], OO858, Euler gives a proof of the pentagonal number theorem. Euler recalls his discovery of a recurrence relation for the divisor function, and that it assumes the pentagonal number theorem, which he had not been able to prove. He then says: “Since that time however I have also found a demonstration of this theorem, which depends on this lemma:

$$\begin{aligned} (1-\alpha)(1-\beta)(1-\gamma)(1-\delta) \text{ etc.} = \\ 1-\alpha-\beta(1-\alpha)-\gamma(1-\alpha)(1-\beta)-\delta(1-\alpha)(1-\beta)(1-\gamma)-\text{etc.}, \end{aligned}$$

whose demonstration is immediate.

Therefore according to this lemma

$$\begin{aligned} (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5) \text{ etc.} = s = \\ 1-x-x^2(1-x)-x^3(1-x)(1-x^2)-x^4(1-x)(1-x^2)(1-x^3)-\text{etc.} \end{aligned}$$

Were it put $s = 1 - x - Axx$, it will be

$$A = 1 - x + x(1-x)(1-x^2) + x^2(1-x)(1-x^2)(1-x^3) + \text{etc.}$$

The factor $1-x$ is expanded everywhere, and it will be

$$\begin{aligned} A = 1 & & -x-x^2(1-xx) & & -x^3(1-xx)(1-x^3)-\text{etc.} \\ & & +x(1-xx) + x^2(1-xx)(1-x^3) & & +x^3(1-xx)(1-x^3)(1-x^4) + \text{etc.} \end{aligned}$$

and then it can be made

$$A = 1 - x^3 - x^5(1-xx) - x^7(1-xx)(1-x^3) - \text{etc.}$$

Were it $A = 1 - x^3 - Bx^5$, it will be

$$B = 1 - xx + x^2(1-xx)(1-x^3) + x^4(1-xx)(1-x^3)(1-x^4) + \text{etc.}$$

The factor $1-xx$ is expanded:

$$\begin{aligned} B = 1 & & -xx-x^4(1-x^3) & & -x^6(1-x^3)(1-x^4)-\text{etc.} \\ & & +xx(1-x^3) + x^4(1-x^3)(1-x^4) & & +x^6(1-x^3)(1-x^4)(1-x^5) + \text{etc.}, \end{aligned}$$

and then it can be made

$$B = 1 - x^5 - x^3(1 - x^3) - x^{11}(1 - x^3)(1 - x^4) - \text{etc.}$$

Were it $B = 1 - x^5 - Cx^8$, it will be

$$C = 1 - x^3 + x^3(1 - x^3)(1 - x^4) + x^6(1 - x^3)(1 - x^4)(1 - x^5) + \text{etc.}$$

The factor $1 - x^3$ is expanded:

$$\begin{aligned} C = 1 & \quad -x^3 - x^6(1 - x^4) & \quad -x^9(1 - x^4)(1 - x^5) - \text{etc.} \\ & +x^3(1 - x^4) + x^6(1 - x^4)(1 - x^5) & +x^9(1 - x^4)(1 - x^5)(1 - x^6) + \text{etc.}, \end{aligned}$$

therefore

$$C = 1 - x^7 - x^{11}(1 - x^4) - x^{15}(1 - x^4)(1 - x^5) - \text{etc.}$$

Were it $C = 1 - x^7 - Dx^{11}$ etc. If one continues in the same manner, then

$$D = 1 - x^9 - Ex^{14}, \quad E = 1 - x^{11} - Fx^{17}, \quad \text{etc.}$$

And in consequence:

$$\left. \begin{aligned} s &= 1 - x - Ax^2 \\ A &= 1 - x^3 - Bx^5 \\ B &= 1 - x^5 - Cx^8 \\ C &= 1 - x^7 - Dx^{11} \\ D &= 1 - x^9 - Ex^{14} \\ &\text{etc.} \end{aligned} \right\} \text{or} \left\{ \begin{aligned} s &= 1 - x - Ax^2 \\ Ax^2 &= x^2(1 - x^3) - Bx^7 \\ Bx^7 &= x^7(1 - x^5) - Cx^{15} \\ Cx^{15} &= x^{15}(1 - x^7) - Dx^{26} \\ Dx^{26} &= x^{26}(1 - x^9) - Ex^{40} \\ &\text{etc.} \end{aligned} \right.$$

from which it doubtlessly follows

$$s = 1 - x - x^2(1 - x^3) + x^7(1 - x^5) - x^{15}(1 - x^7) + x^{26}(1 - x^9) - \text{etc.}$$

or

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - \text{etc.}''$$

In his "Demonstratio theorematis circa ordinem in summis divisorum observatum", published in the *Novi commentarii academiae scientiarum imperialis Petropolitanae* in 1760 [Euler, 1760a], E244, Euler gives the above proof of the pentagonal number theorem from his letter to Goldbach in more detail. Euler first recalls that some time ago he had discovered a recurrence relation for the divisor function, for which the differences in the arguments are the pentagonal numbers, but that his proof was not rigorous enough to him, since it relied on the pentagonal number theorem. But now he declares that he has finally found a demonstration of this, and thus "Not any doubt may remain of this property, the demonstration of the truth of which rests on each of these propositions, which I will now set forth and demonstrate". In Proposition I, Euler notes that for $s = (1 + \alpha)(1 + \beta)(1 + \gamma)(1 + \delta)(1 + \varepsilon)(1 + \zeta)(1 + \eta)$ etc., then

$$\begin{aligned} s &= (1 + \alpha) + \beta(1 + \alpha) + \gamma(1 + \alpha)(1 + \beta) + \delta(1 + \alpha)(1 + \beta)(1 + \gamma) \\ &+ \varepsilon(1 + \alpha)(1 + \beta)(1 + \gamma)(1 + \delta) + \zeta(1 + \alpha)(1 + \beta)(1 + \gamma)(1 + \delta)(1 + \varepsilon) + \text{etc.} \end{aligned}$$

Taking $\alpha = -x$, $\beta = -xx$, $\gamma = -x^3$, $\delta = -x^4$, $\varepsilon = -x^5$, etc., Euler then proves Proposition II, that if $s = (1 - x)(1 - xx)(1 - x^3)(1 - x^4)(1 - x^5)(1 - x^6)$ etc., then

$$s = 1 - x - xx(1 - x) - x^3(1 - x)(1 - x^2) - x^4(1 - x)(1 - x^2)(1 - x^3) - \text{etc.}$$

In Proposition III, Euler then proves the pentagonal number theorem, giving precisely the same proof which he gave in his above letter to Goldbach. Euler then

uses this to finally give a rigorous proof of his recurrence relation for the divisor function, which we discuss in Section 4.

Euler gives two proofs of the pentagonal number theorem in “Evolutio producti infiniti $(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc. in seriem simplicem”, delivered to the St. Petersburg Academy on August 14, 1775 and published in 1783 in the *Acta academiae scientiarum imperialis Petropolitinae* [Euler, 1783b], E541. The first proof in §§1–10 is the same as his earlier proof except that instead of taking $s = 1 - x - Axx$, Euler takes $s = 1 - x - A$, instead of taking $s = 1 - x^3 - Bx^5$, Euler takes $A = xx - x^5 - B$, instead of taking $B = 1 - x^5 - Cx^8$, Euler takes $B = x^7 - x^{12} - C$, etc. In the second proof in §§11–18, having taken $s = (1-x)(1-x^2)(1-x^3)$ etc., Euler states, “Of course then

$$s = 1 - x - xx(1-x) - x^3(1-x)(1-xx) - x^4(1-x)(1-x^2)(1-x^3) - \text{etc.},$$

for which by expanding the second member $-xx(1-x)$, it will be

$$s = 1 - x - xx + x^3 - x^3(1-x)(1-xx) - x^4(1-x)(1-xx)(1-x^3) - \text{etc.},$$

and it is set $s = 1 - x - xx + A$ so that

$$A = x^3 - x^3(1-x)(1-xx) - x^4(1-x)(1-xx)(1-x^3) - \text{etc.},$$

for which, by expanding each term by the factor $1-x$, it is broken into two parts such that it appears as

$$\begin{aligned} A = & x^3 - x^3(1-xx) - x^4(1-xx)(1-x^3) - x^5(1-x^2)(1-x^3)(1-x^4) \dots \\ & + x^4(1-xx) + x^5(1-xx)(1-x^3) + x^6(1-x^2)(1-x^3)(1-x^4) \dots \end{aligned}$$

Here, contracting the pairs of terms with the same powers of x like before will produce this

$$A = +x^5 + x^7(1-xx) + x^9(1-xx)(1-x^3) + x^{11}(1-x^2)(1-x^3)(1-x^4) \text{ etc.}”$$

Euler then expands again the second term, and takes $A = x^5 + x^7 - B$. He expands each term in B then by the factor $1-xx$, and expresses this as for A in a series of terms with negative signs and a series of terms with positive signs, which he then combines to get $B = x^{12} + x^{15}(1-x^3) + x^{18}(1-x^3)(1-x^4) + x^{21}(1-x^3)(1-x^4)(1-x^5) + \text{etc.}$. Euler continues this up to $E = x^{51} + x^{57}(1-x^6) + x^{63}(1-x^6)(1-x^7) + x^{69}(1-x^6)(1-x^7)(1-x^8) + \text{etc.}$, and notes that, “Here indeed the order of the exponents is easily perceived. For with the values of the letters A, B, C, D , at first the first terms were simply $x^3, x^9, x^{18}, x^{30}, x^{45}$, where the exponents are clearly triples of triangular numbers, from which in general the exponent for the number n will be $\frac{3nn+3n}{2}$. However, the differences between two successive powers of x is the same as n , because of which the number n will be subtracted twice from this formula, from which the exponents follow as $\frac{3nn+n}{2}$ and $\frac{3nn-n}{2}$ ”.

We now shall give a formal proof by induction of the pentagonal number theorem that is essentially Euler’s above proof. It will be convenient to write $\omega_n = \frac{3n(n-1)}{2}$. Let $P = \prod_{m=1}^{\infty} (1-x^m) = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5) \dots$. Thus

$$\begin{aligned} P = & 1 - x - (1-x)x^2 - (1-x)(1-x^2)x^3 - (1-x)(1-x^2)(1-x^3)x^4 \\ & - (1-x)(1-x^2)(1-x^3)(1-x^4)x^5 - \dots \end{aligned}$$

Expanding out the factor $1 - x$ in each term of P yields

$$\begin{aligned} P &= 1 - x - x^2 + x^3 - (1 - x^2)x^3 + (1 - x^2)x^4 - (1 - x^2)(1 - x^3)x^4 \\ &\quad + (1 - x^2)(1 - x^3)x^5 - (1 - x^2)(1 - x^3)(1 - x^4)x^5 \\ &\quad + (1 - x^2)(1 - x^3)(1 - x^4)x^6 - + \dots \end{aligned}$$

Adding the coefficients of each power of x gives

$$\begin{aligned} P &= 1 - x - x^2 + x^5 + (1 - x^2)x^7 + (1 - x^2)(1 - x^3)x^9 \\ &\quad + (1 - x^2)(1 - x^3)(1 - x^4)x^6 - + \dots \end{aligned}$$

Since all the following terms have degree ≥ 6 , it is clear that the first three terms of P are $1 - x - x^2$. We now make the induction assumption that up to some k , the first $2k + 1$ terms of P satisfy

$$\begin{aligned} P &= \sum_{n=-k+1}^k (-1)^n x^{\omega_n} \\ &\quad + (-1)^k x^{\omega_k+k} \left((1 - x^k) + (1 - x^k)(1 - x^{k+1})x^k \right. \\ &\quad \left. + (1 - x^k)(1 - x^{k+1})(1 - x^{k+2})x^{2k} + \dots \right) \end{aligned}$$

Clearly this holding for arbitrary k is equivalent to the pentagonal number theorem.

Now expanding out the factor $1 - x^k$ in each term we obtain

$$\begin{aligned} P &= \sum_{n=-k+1}^k (-1)^n x^{\omega_n} \\ &\quad + (-1)^k x^{\omega_k+k} \left(1 - x^k + (1 - x^{k+1})x^k - (1 - x^{k+1})x^{2k} \right. \\ &\quad \left. + (1 - x^{k+1})(1 - x^{k+2})x^{2k} - (1 - x^{k+1})(1 - x^{k+2})x^{3k} + \dots \right) \end{aligned}$$

Adding the coefficients of each power then gives

$$\begin{aligned} P &= \sum_{n=-k+1}^k (-1)^n x^{\omega_n} \\ &\quad + (-1)^k x^{\omega_k+k} \left(1 - x^{2k+1} + (1 - x^{k+1})x^{3k+2} - (1 - x^{k+1})(1 - x^{k+2})x^{3k} + \dots \right) \end{aligned}$$

Therefore we have

$$\begin{aligned} P &= \sum_{n=-k+1}^k (-1)^n x^{\omega_n} + (-1)^k x^{\omega_k+k} + (-1)^{k+1} x^{\omega_k+3k+1} \\ &\quad + (-1)^{k+1} x^{\omega_k+4k+2} \left(1 + (1 - x^{k+1})x^{k+1} + (1 - x^{k+1})(1 - x^{k+2})x^{2k+2} + \dots \right) \end{aligned}$$

But $\omega_k + k = \frac{k(3k-1)+2k}{2} = \frac{k(3k+1)}{2} = \omega_{-k}$ and $\omega_k + 3k + 1 = \frac{k(3k-1)+6k+2}{2} = \frac{(k+1)(3k+2)}{2} = \omega_{k+1}$, hence by replacing k with $k + 1$ we obtain

$$\begin{aligned} P &= \sum_{n=-k}^{k+1} (-1)^n x^{\omega_n} \\ &\quad + (-1)^{k+1} x^{\omega_{k+1}+k+1} \left(1 + (1 - x^{k+1})x^{k+1} + (1 - x^{k+1})(1 - x^{k+2})x^{2k+2} + \dots \right) \end{aligned}$$

which completes the induction, thus proving the pentagonal number theorem.

4. EULER'S RECURRENCE RELATION FOR THE DIVISOR FUNCTION

We recall the divisor function $\int n$, the sum of all the divisors of n , including itself. Euler first discussed his recurrence relation for the divisor function in a letter to Goldbach on April 1, 1747, Letter 113 in the Euler-Goldbach correspondence [Juškevič and Winter, 1965], OO827. He begins the letter by saying, "I have recently discovered a very amazing order in the integers, which the sums of the divisors of the natural numbers present, which appeared so much more peculiar to me, since in this a great connection with the order of the prime numbers appears to hide. Therefore I ask for some attention to this.

If n denotes any particular integral number, then $\int n$ should denote the sum of all the divisors of this number b . Therefore we have:

$$\begin{array}{l} \int 1 = 1 \\ \int 2 = 1 + 2 = 3 \\ \int 3 = 1 + 3 = 4 \\ \int 4 = 1 + 2 + 4 = 7 \\ \int 5 = 1 + 5 = 6 \\ \int 6 = 1 + 2 + 3 + 6 = 12 \\ \int 7 = 1 + 7 = 8 \\ \int 8 = 1 + 2 + 4 + 8 = 15 \end{array} \left| \begin{array}{l} \int 9 = 1 + 3 + 9 = 13 \\ \int 10 = 1 + 2 + 5 + 10 = 18 \\ \int 11 = 1 + 11 = 12 \\ \int 12 = 1 + 2 + 3 + 4 + 6 + 12 = 28 \\ \int 13 = 1 + 13 = 14 \\ \int 14 = 1 + 2 + 7 + 14 = 24 \\ \int 15 = 1 + 3 + 5 + 15 = 24 \\ \int 16 = 1 + 2 + 4 + 8 + 16 = 31 \\ \text{etc.} \end{array} \right.$$

Given this meaning for the symbol \int , I have found that

$$\int n = \int(n-1) + \int(n-2) - \int(n-5) - \int(n-7) + \int(n-12) + \int(n-15) - \int(n-22) - \int(n-26) + \text{etc.},$$

where always the two signs $+$ and $-$ follow themselves. The order of the derived numbers 1, 2, 5, 7, 12, 15, etc. appears from their differences, and if the same alternations are considered, one immediately sees that,

	1,	2,	5,	7,	12,	15,	22,	26,	35,	40,
Diff.	1,	3,	2,	5,	3,	7,	4,	9,	5,	11,
	51,	57,	70,	77,	92,	100,	117,	126,	145,	etc.
Diff.	6,	13,	7,	15,	8,	17,	9,	19,	etc.	

Furthermore it should be noted that in each case one needs not take more terms once the negative numbers are come to, and if such a term $\int 0$ appears, then for this the given number n must be written, so that in such a case $\int 0 = n$. The

following examples will illuminate the truth of this theorem:

- When then it will be
1. $n = 1;$ $\int 1 = \int 0 = 1$
 2. $n = 2;$ $\int 2 = \int 1 + \int 0 = 1 + 2 = 3$
 3. $n = 3;$ $\int 3 = \int 2 + \int 1 = 3 + 1 = 4$
 4. $n = 4;$ $\int 4 = \int 3 + \int 2 = 4 + 3 = 7$
 5. $n = 5;$ $\int 5 = \int 4 + \int 3 - \int 0 = 7 + 4 - 5 = 6$
 6. $n = 6;$ $\int 6 = \int 5 + \int 4 - \int 1 = 6 + 7 - 1 = 12$
 7. $n = 7;$ $\int 7 = \int 6 + \int 5 - \int 2 - \int 0 = 12 + 6 - 3 - 7 = 8$
 8. $n = 8;$ $\int 8 = \int 7 + \int 6 - \int 3 - \int 1 = 8 + 12 - 4 - 1 = 15$
 9. $n = 9;$ $\int 9 = \int 8 + \int 7 - \int 4 - \int 2 = 15 + 8 - 7 - 3 = 13$
 10. $n = 10;$ $\int 10 = \int 9 + \int 8 - \int 5 - \int 3 = 13 + 15 - 6 - 4 = 18$
 11. $n = 11;$ $\int 11 = \int 10 + \int 9 - \int 6 - \int 4 = 18 + 13 - 12 - 7 = 12$
 12. $n = 12;$ $\int 12 = \int 11 + \int 10 - \int 7 - \int 5 + \int 0 = 12 + 18 - 8 - 6 + 12 = 28$
- etc.

The reason for this order is not obvious, since one does not see how the numbers 1, 2, 5, 7, 12, 15, etc. relate with the nature of the divisors. I can also not claim that I have been able to give a rigorous proof of this either. However, if I had no proof at all, one would still not be able to doubt the truth of it, because over 300 cases always follow this rule. In the mean time, I have correctly derived this theorem from the following statement.

If $s = (1 - x)(1 - x^2)(1 - x^3)(1 - x^4)(1 - x^5)$ etc., then also $s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} +$ etc., where the exponents of x are the same numbers which appeared earlier on; and if this statement is true, which I do not doubt, despite the fact that I do not have a rigorous demonstration, then the theorem is completely justified.

For from the double values of s , I obtain firstly

$$\frac{ds}{s} = \frac{-dx}{1-x} - \frac{2xdx}{1-x^2} - \frac{3x^2dx}{1-x^3} - \frac{4x^3dx}{1-x^4} - \frac{5x^4dx}{1-x^5} - \text{etc.}$$

and then

$$\frac{ds}{s} = \frac{-dx - 2xdx + 5x^4 + 7x^6 - 12x^{11} - 15x^{14} + \text{etc.}}{1 - x - xx + x^5 + x^7 - x^{12} - x^{15} + \text{etc.}}$$

therefore we have

$$\frac{1}{1-x} + \frac{2x}{1-x^2} + \frac{3x^2}{1-x^3} + \frac{5x^4}{1-x^5} + \frac{4x^3}{1-x^4} + \frac{6x^5}{1-x^6} + \text{etc.} = \frac{1 + 2x - 5x^4 - 7x^6 + 12x^{11} + 15x^{14} - \text{etc.}}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \text{etc.}}$$

If however all of the last pieces are transformed into geometric progressions, then one obtains for the same

$$\begin{array}{cccccccccccccccc}
 1 & +x & +x^2 & +x^3 & +x^4 & +x^5 & +x^6 & +x^7 & +x^8 & +x^9 & +x^{10} & +x^{11} & +x^{12} & +\text{etc.} \\
 & +2x & & +2x^3 & & +2x^5 & & +2x^7 & & +2x^9 & & +2x^{11} & & \\
 & & +3x^2 & & & +3x^5 & & & +3x^8 & & & +3x^{11} & & \\
 & & & +4x^3 & & & & +4x^7 & & & & +4x^{11} & & \\
 & & & & +5x^4 & & & & & +5x^9 & & & +6x^{11} & \\
 & & & & & +6x^5 & & & & & & & & \\
 & & & & & & +7x^6 & & & & & & & \\
 & & & & & & & +8x^7 & & & & & & \\
 & & & & & & & & +9x^8 & & & & & \\
 & & & & & & & & & +10x^9 & & & & \\
 & & & & & & & & & & +11x^{10} & & & \\
 & & & & & & & & & & & +12x^{11} & & \\
 & & & & & & & & & & & & +13x^{12} & \\
 & & & & & & & & & & & & & +\text{etc.}
 \end{array}$$

that is:

$$\begin{aligned}
 1 + \int 2x + \int 3x^2 + \int 4x^3 + \int 5x^4 + \int 6x^5 + \int 7x^6 + \int 8x^7 + \int 9x^8 + \int 10x^9 + \text{etc.} \\
 = \frac{1 + 2x - 5x^4 - 7x^6 + 12x^{11} + 15x^{14} - 22x^{21} - 226x^{25} + 35x^{34} + \text{etc.}}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - \text{etc.}},
 \end{aligned}$$

from which the given theorem easily ensues. One sees however at the same time, that this is not very obvious, and that without a doubt there are further beautiful things hidden within this.”

The first paper in which Euler gives his recurrence relation for the divisor function is his “Découverte d’une loi tout extraordinaire des nombres, par rapport à la somme de leurs diviseurs”, presented to the Berlin Academy on June 22, 1747, and published in the *Bibliothèque impartiale* in 1751 [Euler, 1751a], E175. Euler proves this recurrence relation in the same way as he did in his April 1, 1747 letter to Goldbach, which we explained above.

Euler writes next to d’Alembert about the pentagonal number theorem in a letter on February 15, 1748, Letter 13 in the Euler-d’Alembert correspondence in the *Opera Omnia* [Juškevič and Taton, 1980], OO25. Euler writes: “Regarding this series that I spoke to you about, I found from it a very peculiar property about numbers with respect to the sum of the divisors of each number. That $\int n$ represents the sum of all the divisors of n so that $\int 1 = 1$; $\int 2 = 3$; $\int 3 = 4$; $\int 4 = 7$; $\int 5 = 6$; $\int 6 = 12$; $\int 7 = 8$ etc. it seemed initially almost impossible to discover any law in the sequence of numbers, but I found that each term depends on some of the previous ones, according to this formula:

$$\int n = \int(n-1) + \int(n-2) - \int(n-5) - \int(n-7) + \int(n-12) + \int(n-15) - \int(n-22) - \text{etc.}$$

where it is worthy of note 1° that the numbers

$$\begin{array}{cccccccccccc}
 1, & 2, & 5, & 7, & 12, & 15, & 22, & 26, & 35, & 40, & \text{etc.} \\
 1 & 3 & 2 & 5 & 3 & 7 & 4 & 9 & 5 & &
 \end{array}$$

are easily obtained by the differences considered alternately. 2° In each case we only take the numbers where the number after the \int sign are non-negative. 3° If we obtain the term $\int 0$ or $\int(n - n)$ we will take n as the value.

Thus you will see that

$$\begin{aligned} \int 4 &= \int 3 + \int 2 = 7; \\ \int 9 &= \int 8 + \int 7 - \int 4 - \int 2 = 15 + 8 - 7 - 3 = 13; \\ \int 15 &= \int 14 + \int 13 - \int 10 - \int 8 + \int 3 + \int 0 = 24 + 14 - 18 - 15 + 4 + 15 = 24; \\ \int 35 &= \int 34 + \int 33 - \int 30 - \int 28 + \int 23 + \int 20 - \int 13 - \int 9 + \int 0 = \\ &54 + 48 - 72 - 56 + 24 + 42 - 14 - 13 + 35 = 48. \end{aligned}$$

So every time that n is a prime number we will find that $\int n = n + 1$ and since the nature of prime numbers enters this investigation, this law seems to me even more remarkable.”

d’Alembert replies to Euler in a letter on March 30, 1748, Letter 14 in the Euler-d’Alembert correspondence in the *Opera omnia* [Juškevič and Taton, 1980], OO26, saying, “That Sir, what comes to my mind while writing you, I only have room left to say that your theorem on series seems very beautiful”.

On April 6, 1752, Euler presented his paper “Observatio de summis divisorum” to the St. Petersburg Academy, published in 1760 in the *Novi commentarii academiae scientiarum imperialis Petropolitanae* [Euler, 1760b], E243. This paper just repeats what Euler said in the letter to Goldbach and paper we considered above.

In the same volume of this journal, Euler published the paper “Demonstratio theorematis circa ordinem in summis divisorum observatum” [Euler, 1760a], E244, which again relates this recurrence relation for the divisor function (although more clearly), and also gives an inductive proof of the pentagonal number theorem, which we discussed in Section 3.

Goldbach writes a letter to Euler on May 9, 1752, Letter 157 in the Euler-Goldbach correspondence [Juškevič and Winter, 1965], OO871, in which he says that (Augustin Nathaniel) Grischow has written to him about the presentation Euler gave to the St. Petersburg Academy on the sums of divisors. Goldbach writes, “I find myself at the present time not in the position to judge, only you have insight into such matters, though let me have no doubt about the truth of everything that has been said in this dissertation. In particular I saw with great pleasure that in the numbers 1, 2, 5, 7, 12, 15, 22, etc. was such a beautiful order as was remarked before”.

5. EULER’S RECURRENCE RELATION FOR THE PARTITION FUNCTION

We recall the partition function $n^{(m)}$, the number of ways of expressing n as a sum of positive integers less than or equal to m , disregarding order. The only work in which Euler mentions his pentagonal number recurrence relation for the partition function (there is no extant correspondence that discusses it) is his paper “De partitione numerorum”, presented to the St. Petersburg Academy on January 26, 1750 and published in the *Novi commentarii academiae scientiarum imperialis Petropolitanae* in 1753 [Euler, 1753], E191. In §40, Euler says, “Certainly it is manifest from the nature of this matter for it truly to be a recurrent series, with it

arising from the expansion of this fraction:

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6) \text{ etc.}}$$

Therefore there will be a ladder relation for this series, if the denominator were expanded by multiplication. Indeed with this multiplication having been carried out, the denominator will be found to be expressed in the following way:

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{52} + x^{57} - x^{70} - x^{77} + \text{etc.}$$

These powers of x hold to such a rule, from which formation is seen to be able to be determined with difficulty; in the meantime however, from inspection it is soon apparent for the pairs of terms to alternately be positive and negative. No less the exponents of x are observed to hold to a certain law, from which the general term is gathered to be $x^{\frac{n(3n\pm 1)}{2}}$. Namely no other powers occur aside from those whose exponents are contained in the form $\frac{3nn\pm n}{2}$, and for which the powers which arise from n taken as an odd number have the sign $-$, and indeed those formed from an even number, the sign $+$."

Then in §41, Euler continues, "This form therefore provides to us the ladder of relation for the series which has been found, which comes out to be

$$\begin{aligned} n^{(\infty)} = & (n-1)^{(\infty)} + (n-2)^{(\infty)} - (n-5)^{(\infty)} - (n-7)^{(\infty)} + (n-12)^{(\infty)} \\ & + (n-15)^{(\infty)} - (n-22)^{(\infty)} - (n-26)^{(\infty)} + (n-35)^{(\infty)} + (n-40)^{(\infty)} \\ & - (n-51)^{(\infty)} - (n-57)^{(\infty)} + \text{etc.} \end{aligned}$$

Indeed by trying this rule of the progression its place will easily be able to be obtained. For were it $n = 30$ it will be found to be:

$$30^{(\infty)} = 29^{(\infty)} + 28^{(\infty)} - 25^{(\infty)} - 23^{(\infty)} + 18^{(\infty)} + 15^{(\infty)} - 8^{(\infty)} - 4^{(\infty)}$$

which indeed with these numbers taken from the table

$$5604 = 4565 + 3718 - 1958 - 1255 + 385 + 176 - 22 - 5.$$

And indeed in this way it will be pleasant for such series to always be continued".

6. DIVERGENT SERIES OF THE PENTAGONAL NUMBERS

The last paper in which Euler discusses the pentagonal number theorem and its applications is his "De mirabilibus proprietatibus numerorum pentagonalium", presented to the St. Petersburg Academy on September 4, 1775, and published in the *Acta academiae scientiarum imperialis Petropolitinae* in 1783 [Euler, 1783a], E542. In §2, Euler notes that every pentagonal number is one-third of a triangular number (those numbers in the form $\frac{n(n+1)}{2}$), and in §4 he gives his recurrence relation for the divisor function. Then in §7, Euler states the pentagonal number theorem, remarking that, "This then deserves our admiration no less than the properties mentioned above, with no fixed rule apparent from which any connection can be understood between the expansion of this product and our pentagonal numbers".

In the rest of this paper, Euler considers the summation of series of the pentagonal numbers. (We recall from Hans Rademacher's "Comments on Euler's 'De

mirabilibus proprietatibus numerorum pentagonalium' ” [Rademacher, 1969] that the Euler method of summation of a series $\sum_{k=1}^{\infty} a_k$ is defined by:

$$\sum_{k=1}^{\infty} a_k = \lim_{N \rightarrow \infty} \sum_{m=1}^N \left(\frac{1}{2}\right)^{m+1} \sum_{n=0}^m \binom{m}{n} a_n.$$

We find (cf. Rademacher [Rademacher, 1969]) that by this summation, all the sums of the divergent series which Euler gives in this paper may be rigorously justified.)

In §8, Euler states that “Therefore with this series of powers of x equal to this infinite product, if it were set equal to nothing, so that we have this equation:

$$0 = 1 - x^1 - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{etc.}$$

it will involve all the roots, which the product equated to nothing includes”. Then in §9, Euler says, “It is then clear for all the roots of each power from unity to simultaneously be equal to the roots of our equation”, and with $x^n = \cos 2i\pi \pm \sqrt{-1} \sin 2i\pi$, “if for n and i are taken all the successive integral numbers, the formula $x = \cos \frac{2i\pi}{n} \pm \sqrt{-1} \frac{2i\pi}{n}$ will produce all the roots of our equation”.

In §10, Euler states that for $\alpha, \beta, \gamma, \delta, \varepsilon$, etc. the roots of unity, “we gather for the sum of all these fractions to be $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} + \text{etc.} = 1$, the sum of the products from two to be equal to -1 , then indeed the sum of the products from three to be equal to 0 , the sum of the products from four equal to 0 , the sum of the products from five equal to -1 , the sum of the products from six equal to 0 , the sum of the products from seven equal to -1 , etc. Then indeed we can also deduce the sum of all the squares of these fractions, namely

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} + \text{etc.} = 3,$$

the sum of the cubes

$$\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3} + \text{etc.} = 4,$$

the sum of the biquadrates

$$\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4} + \text{etc.} = 7,$$

and so on thusly, where however no order is obvious”.

In §§11-13, Euler notes a few other properties of roots of unity: the reciprocal of a root of unity is itself a root of unity; an n th root of unity α satisfies $\alpha^{in+\lambda} = \alpha^\lambda$ for i an integer; and, roots of roots of unity are themselves roots of unity.

In §14, Euler writes that “With us having assumed here α to be a root of the equation $1 - x^n = 0$, we may run through the cases successively in which n is 1, 2, 3, 4, etc. And foremost, if $n = 1$ it is necessarily $\alpha = 1$, with which value having been substituted into our general equation this form is induced:

$$1 - 1 - 1 + 1 + 1 - 1 - 1 + 1 + \text{etc.}$$

which series is manifestly conflated from infinitely many periods, each of which contains the terms $1 - 1 - 1 + 1$, from which the value of each period is equal to 0 , and thus the infinitely many periods which have been taken at once have a sum equal to 0 ”. In §15, Euler gives an alternate justification of this with the sum of the Leibnitz (i.e. alternating) series $1 - 1 + 1 - 1 + 1 - 1 + \text{etc.} = \frac{1}{2}$.

In §16, Euler writes “We may now consider the case in which $n = 2$ and $\alpha\alpha = 1$, in which indeed α is either $+1$ or -1 . We shall retain the letter α for designating either one of these however, and with

$$\alpha^3 = \alpha, \alpha^4 = 1, \alpha^5 = \alpha, \alpha^6 = 1, \text{ etc.}$$

having been substituted into our general equation this form will be induced:

$$1 - \alpha - 1 + \alpha + \alpha - 1 - \alpha - 1 \mid + 1 - \alpha - 1 + \alpha + \alpha - 1 - \alpha + 1 \text{ etc.}$$

which series progresses evenly by certain periods, which are replicated continuously, and each of which is composed of these eight terms:

$$1 - \alpha - 1 + \alpha + \alpha - 1 - \alpha + 1,$$

of which the sum is 0, and thus is certain to vanish however large a number of integral periods”. Then in §17, Euler states that the this series can be separated into two subseries which must both be equal to 0:

$$\begin{aligned} 1 - 1 - 1 + 1, +1 - 1 - 1 + 1, +1 - 1 - 1 + 1, \text{ etc.} &= 0, \\ -\alpha + \alpha + \alpha - \alpha, -\alpha + \alpha + \alpha - \alpha, -\alpha + \alpha + \alpha - \alpha, \text{ etc.} &= 0. \end{aligned}$$

In §§18 and 19, Euler considers the cases $\alpha^3 = 1$ and $\alpha^4 = 1$, getting

$$\begin{aligned} -\alpha^2 + \alpha^2 + \alpha^2 - \alpha^2, -\alpha^2 + \alpha^2 + \alpha^2 - \alpha^2, -\alpha^2 + \alpha^2 + \alpha^2 - \alpha^2, \text{ etc.} &= 0, \\ +\alpha^3 - \alpha^3 - \alpha^3 + \alpha^3, +\alpha^3 - \alpha^3 - \alpha^3 + \alpha^3, +\alpha^3 - \alpha^3 - \alpha^3 + \alpha^3, \text{ etc.} &= 0. \end{aligned}$$

In §20, Euler examines $\alpha^5 = 1$, writing, “here not all the powers less than five occur. If it were $\alpha^5 = 1$, this periodic series will be produced:

$$\begin{array}{l} 1 - \alpha + 1 - \alpha^2 + \alpha^2 - 1 + \alpha - 1 + \alpha^2 - \alpha^2 + 1 \mid \\ -\alpha^2 + \alpha^2 - 1 + \alpha - 1 + \alpha^2 - \alpha^2 + 1 - \alpha \end{array} \left| \begin{array}{l} -\alpha + 1 \\ +1 - \alpha^2 + \alpha^2 \end{array} \right. \text{ etc.}$$

where the powers α^3 and α^5 are entirely excluded”. Then, in §§20–22, Euler observes that for all higher roots of unity the series are composed from periods which each sum to zero, and in §23 that if α is an n th root of unity so that $1 - \frac{x}{\alpha}$ is a factor of $1 - x^n$, then $1 - \frac{x}{\alpha}$ is also a factor of $1 - x^{2n}, 1 - x^{3n}, 1 - x^{4n}$, etc., and thus for each root of unity to have infinite multiplicity in the equation $\sum_{n=0}^{\infty} x^{\frac{3x^2 \pm x}{2}} = 0$.

Euler considers divergent series of the pentagonal numbers in §§24–31 of the paper. In §24, he writes: “We know moreover from the nature of equations, if an arbitrary equation

$$1 + Ax + Bxx + Cx^3 + Dx^4 + \text{etc.} = 0,$$

should have two roots equal to α , then in fact for α to be a root of the equation born from differentiating, namely:

$$A + 2Bx + 3Cxx + 4Dx^3 + \text{etc.} = 0,$$

and if it has three roots equal to α , then in addition α will also be a root of the equation born from differentiating after indeed we multiply this differentiated equation by x

$$1^2 \cdot A + 2^2 \cdot Bx + 3^2 \cdot Cxx + 4^2 \cdot Dx^3 + \text{etc.} = 0,$$

from which if this equation were to have λ equal roots, each of which were equal to α , then it will always be

$$1^\lambda \cdot A + 2^\lambda \cdot B\alpha + 3^\lambda \cdot C\alpha^3 + 4^\lambda \cdot D\alpha^4 + \text{etc.} = 0,$$

from which if for the grace of uniformity we were to multiply this equation by α , it will then be

$$1^\lambda \cdot A\alpha + 2^\lambda \cdot B\alpha^2 + 3^\lambda \cdot C\alpha^3 + 4^\lambda \cdot D\alpha^4 + \text{etc.} = 0''.$$

In §25, Euler writes: “Therefore by putting $\alpha^n = 1$ our equation formed from the pentagonal numbers

$$1 - x^1 - x^2 + x^5 + x^7 - x^{12} - x^{15} + \text{etc.} = 0,$$

shall have infinitely many roots equal to α , and thus α will be a root of all equations contained in this general form:

$$-1^\lambda x - 2^\lambda x^2 + 5^\lambda x^5 + 7^\lambda x^7 - 12^\lambda x^{12} - \text{etc.} = 0$$

for any integer whatsoever taken for λ . Therefore it will always be

$$-1^\lambda \alpha - 2^\lambda \alpha^2 + 5^\lambda \alpha^5 + 7^\lambda \alpha^7 - 12^\lambda \alpha^{12} - \text{etc.} = 0''.$$

In §26, Euler then writes, “In order to make this clear, we shall take $\alpha = 1$, and it will always be

$$-1^\lambda - 2^\lambda + 5^\lambda + 7^\lambda - 12^\lambda - 15^\lambda + \text{etc.} = 0,$$

and for the case $\lambda = 0$ we have already probed the truth of this equation. Were it therefore $\lambda = 1$, it will be revealed for the sum of this series diverging to infinity:

$$-1 - 2 + 5 + 7 - 12 - 15 + 22 + 26 - \text{etc.}$$

to be equal to 0. Seeing moreover that this series is broken up, that is, it is interpolated from two series, each of which may be contemplated individually, by putting

$$\begin{aligned} s &= -1 + 5 - 12 + 22 - 35 + \text{etc.} \text{ and} \\ t &= -2 + 7 - 15 + 26 - 40 + \text{etc.} \end{aligned}$$

where it ought to be shown to become $s + t = 0$ ". In §27 Euler continues, “Indeed from the doctrine of series, which proceed with alternating signs, such as $A - B + C - D + \text{etc.}$, it is known for the sum of this series progressing into infinity to be equal to $\frac{1}{2}A - \frac{1}{4}(B - A) + \frac{1}{8}(C - 2B + A) - \frac{1}{16}(D - 3C + 3B - A) \text{etc.}$, which rule is thus conveniently related by differences, namely by a rule based on the signs. From the series of numbers $A, B, C, D, E, \text{etc.}$ should be formed a series of differences, so that each term of this series is subtracted from the following one: it will be $a, b, c, d, \text{etc.}$ Again, by the same law, from this series of differences should be formed the series of second differences, which shall be $a', b', c', d', \text{etc.}$, from this series in turn the series of third differences, which shall be $a'', b'', c'', d'', e'', \text{etc.}$, and thus in this way beyond, until constant differences prevail. Then moreover from the first terms of all these series the sum of the proposed series may thus be determined, such that it will be

$$\frac{1}{2}A - \frac{1}{4}a + \frac{1}{8}a' - \frac{1}{16}a'' + \frac{1}{32}a''' - \frac{1}{64}a'''' + \text{etc.}''$$

In §28, Euler then writes, “With this rule which has been established, with the signs having been switched it will be

$$\begin{aligned} -s &= 1 - 5 + 12 - 22 + 35 - 51 + 70 - \text{etc.} \text{ and} \\ -t &= 2 - 7 + 15 - 26 + 40 - 57 + 77 - \text{etc.} \end{aligned}$$

These terms may be arranged in the following way and their differences may be written:

$$\begin{array}{cccccccc}
 1, & 5, & 12, & 22, & 35, & 51, & 70, & \text{etc.} \\
 & 4, & 7, & 10, & 13, & 16, & 19 & \\
 & & 3, & 3, & 3, & 3, & 3, & \\
 & & & 0, & 0, & 0, & 0, & \\
 \end{array} \parallel \begin{array}{cccccccc}
 2, & 7, & 15, & 26, & 40, & 57, & 77, & \text{etc.} \\
 & 5, & 8, & 11, & 14, & 17, & 20 & \\
 & & 3, & 3, & 3, & 3, & 3, & \\
 & & & 0, & 0, & 0, & 0, &
 \end{array}$$

Then it is therefore gathered to be

$$\begin{aligned}
 -s &= \frac{1}{2} - \frac{4}{4} + \frac{3}{8} = -\frac{1}{8}, \text{ that is } s = \frac{1}{8}, \text{ and in turn} \\
 -t &= \frac{2}{2} - \frac{5}{4} + \frac{3}{8} = \frac{1}{8}, \text{ that is } t = -\frac{1}{8}
 \end{aligned}$$

from which it is clearly concluded to be $s + t = 0$ ”.

In §29, Euler writes, “Although the rules by which these properties are supported clearly leave no doubt, it will by no means be useless to exhibit further the truth for the case $\lambda = 2$ for it to be

$$-1^2 - 2^2 + 5^2 + 7^2 - 12^2 - 15^2 + 22^2 + \text{etc.} = 0.$$

Again this series is parted into two, which will be with the sign changed:

$$\begin{aligned}
 s &= +1^2 - 5^5 + 12^2 - 22^2 + 35^2 - 51^2 + \text{etc.} \\
 t &= 2^2 - 7^2 + 15^2 - 26^2 + 40^2 - 57^2 + \text{etc.}
 \end{aligned}$$

and to find the sum of the first, the following operation is instituted:

Series	1,	24,	144,	484,	1225,	2601,	4900
Diff. I		24,	119,	340,	741,	1376,	2299
Diff. II			95,	221,	401,	635,	923
Diff. III				126,	180,	234	288
Diff. IV					54,	54,	54
Diff. V						0,	0

Then it will therefore be

$$s = \frac{1}{2} - \frac{24}{4} + \frac{95}{8} - \frac{126}{16} + \frac{54}{32} = +\frac{3}{16}.$$

In a similar way for the other series

Series	4,	49,	225,	676,	1600,	3249,	5929
Diff. I		45,	176,	451,	924,	1649,	2680
Diff. II			131,	275,	473,	725,	1031
Diff. III				144,	198,	252,	306
Diff. IV					54,	54,	54
Diff. V						0,	0

Then it may be concluded

$$t = \frac{4}{2} - \frac{45}{4} + \frac{131}{8} - \frac{144}{16} + \frac{54}{32} = -\frac{3}{16}.$$

From this it prevails for the total sum to become $s + t = 0$.”

In §30, “We shall now consider as well the square roots, that is when it is $\alpha^2 = 1$, and then such a series will arise:

$$-1^\lambda \cdot \alpha - 2^\lambda + 5^\lambda \cdot \alpha - 7^\lambda \cdot \alpha - 12^\lambda - 15^\lambda \cdot \alpha + 22^\lambda + 26^\lambda - \text{etc.} = 0,$$

from which if we separate the terms containing unity and α from each other, we shall obtain two series equal to nothing, namely:

$$-2^\lambda - 12^\lambda + 22^\lambda + 26^\lambda - 40^\lambda - 70^\lambda + 92^\lambda + \text{etc.} = 0$$

and

$$-1^\lambda \cdot \alpha + 5^\lambda \cdot \alpha + 7^\lambda \cdot \alpha - 15^\lambda \cdot \alpha - 35^\lambda \cdot \alpha + 51^\lambda \cdot \alpha + 57^\lambda \cdot \alpha - \text{etc.} = 0.$$

If indeed we want to display the truth of these series in the same way which we did before, each ought to be divided into four others, until in the end we reach constant differences. And indeed, if this work were undertaken, it will be able to be certain, for the aggregate of all the parts to be equal to 0”.

In §31, Euler says, “Now most generally the total problem is embraced, and it may be $\alpha^n = 1$, and we shall search for the series which contains all the powers α^r . To this end, from all our pentagonal numbers we shall pick out those which divided by n leave the very residue r . Therefore were these pentagonal numbers A, B, C, D, E , etc., namely all those of the form $\gamma n + r$, and the sign of which \pm , which will agree with these, may be noted with care. Then indeed it will always be

$$\pm A^\lambda \pm B^\lambda \pm C^\lambda \pm D^\lambda \pm \text{etc.} = 0,$$

where any integral value may be taken for the exponent λ . Then all the series which we have elicited so far, and of which we have shown for the sums to be equal to nothing, are contained in this most general form.”

7. CONCLUSIONS

As we noted in in Section 1, the pentagonal number theorem is a special case of the Jacobi triple product identity. Using the Jacobi triple product identity we can also obtain an identity for the cube of the product in (1), $\prod_{m=1}^{\infty} (1 - x^m)^3 = \sum_{n=0}^{\infty} (-1)^n (2n + 1) x^{\frac{n(n+1)}{2}}$. C. G. J. Jacobi observes this in his article “Note sur les fonctions elliptiques” in 1828 [Jacobi, 1828], saying that he always found Euler’s pentagonal number theorem “a very surprising and admirable feat”. Indeed, the pentagonal number theorem is one of the first results in the theory of q -series and theta functions.

Euler considered the series expansion of more general infinite products such as $(1 + mz)(1 + m^2z)(1 + m^3z)(1 + m^4z)$ etc. (for example in his November 10, 1742 letter to Niklaus I Bernoulli, his April 5, 1746 letter to Goldbach, and §18 of “Observationes analyticae variae de combinationibus” [Euler, 1751b], which we discussed in Section 2), but he does not seem to have found the Jacobi triple product identity.

In terms of the product representation of theta functions, the pentagonal number theorem is $\vartheta_4(q/6; q) = (q; q)_{\infty}$ [Rademacher, 1973, §78, Chapter 10], for $q^{2/3} = x$.

It would be good to further explore Euler’s work on partitions. Kiselev and Matvievskaia discuss [Kiselev and Matvievskaia, 1965] unpublished notes of Euler on partitions.

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