Lecture 0

§0. Notations. - $\mathbb{Z}$ = group (or ring) of integers.

$\mathbb{Q}$ = field of rational numbers.

$\mathbb{R}$ = field of real numbers $\qquad$ $\mathbb{C}$ = field of complex numbers

$$\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$; here i (iota) is a square root of -1.

Remarks. - (1) $\mathbb{R}$ is an ordered field - i.e., there is a total ordering relation $<$ satisfying :

$$a < b \Rightarrow a + c < b + c$$

$$\begin{matrix} a < b \\ x > 0 \end{matrix} \Rightarrow ax < bx$$

$$\begin{matrix} a < b \\ x < 0 \end{matrix} \Rightarrow bx < ax$$

Thus, in an ordered field, $x^2 > 0$, for every non-zero x.

$\mathbb{C}$ is, hence, NOT an ordered field.

(2) There is no intrinsic difference between i and $-i$, They are both solutions of $T^2 + 1 = 0$. Often people say, "let us choose a solution of $T^2 + 1 = 0$ and call it i".

More precisely $\quad z = x + iy \longmapsto \bar{z} = x - iy$ (complex conjugate) is a field isomorphism which is identity on $\mathbb{R}$.

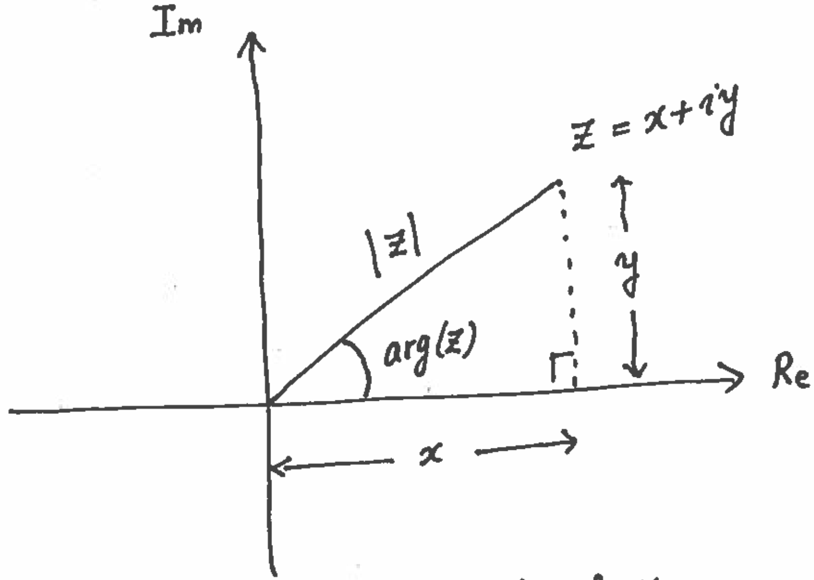
For $z = x + iy \in \mathbb{C}$; $\quad x = \mathrm{Re}(z)$ - real part of z.

$y = \mathrm{Im}(z)$ - imaginary part of z.

§1. Modulus and argument. For $z = x + iy \in \mathbb{C}$,

$|z| := \sqrt{x^2+y^2}$ called modulus (or length, or norm) of z

$\theta = \arg(z)$ called argument, or phase of z – defined only for $z \neq 0$ – is uniquely determined, (up to $2\pi\mathbb{Z}$) by:



$$\cos(\theta) = \frac{x}{\sqrt{x^2+y^2}}$$

$$\sin(\theta) = \frac{y}{\sqrt{x^2+y^2}}$$

(x, y) : Cartesian coord. of z

$(|z|, \arg(z))$: polar coord. of z (if $z \neq 0$).

Very important: Argument of a non-zero complex number is only defined up to $2\pi\mathbb{Z}$. For definiteness, we often take

$$\arg : \mathbb{C} \setminus \{0\} \longrightarrow (-\pi, \pi]$$

but we must keep in mind that this is one of infinitely many choices we could have made.

Note: $|z|^2 = z \cdot \bar{z}$, so, for $z \neq 0$, $\frac{1}{z} = z^{-1} = \frac{\bar{z}}{|z|^2}$.

Using $|\cdot| : \mathbb{C} \to \mathbb{R}_{\geq 0}$, we get the usual Euclidean distance (or metric) on the complex plane.

The usual topological notions of open, closed, compact etc. subsets of $\mathbb{C}$ are defined using this metric.

$$|z - w| = \text{distance between } z, w.$$

§2. Functions of a complex variable. Let $\Omega \subset \mathbb{C}$ be an open and connected subset. Recall the following topological notions.

A set $U \subset \mathbb{C}$ is connected if for any two disjoint open sets $V_1, V_2 \subset \mathbb{C}$ such that $U \subset V_1 \sqcup V_2$, we have: either $U \subset V_1$ or $U \subset V_2$.

We say U is path-connected if for any two points $z_0, z_1 \in U$ there exists a continuous $\gamma : [0,1] \to U$ such that $\gamma(0) = z_0$ and $\gamma(1) = z_1$.

For open subset $U \subset \mathbb{C}$, these two notions are equivalent

We will be studying functions $\boxed{f : \Omega \to \mathbb{C}}$. Such a $\mathbb{C}$-valued function is completely prescribed by its real and imaginary parts

$$f(x + iy) = u(x,y) + i\, v(x,y)$$

where $u, v : \Omega \to \mathbb{R}$ are real-valued functions of two real variables.

The definition of continuity for such functions (even (real-) differentiability) amounts to the same for u and v.

That is, f is continuous at $z_0 = x_0 + iy_0 \in \Omega \iff u$ and v are continuous at (x_0, y_0).

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad , \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \quad \text{etc.}$$

$f \in C^k(\Omega;\mathbb{C}) \iff u, v \in C^k(\Omega;\mathbb{R})$, where, recall that for $u:\Omega \to \mathbb{R}$, $u \in C^k(\Omega;\mathbb{R})$ means all partial derivatives ($k \in \mathbb{Z}_{\geq 0}$) of u, up to order k, exist and are continuous.

e.g. $u \in C^2(\Omega)$ means $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial^2 u}{\partial y^2}$ all exist, and are continuous. Recall - (Clairaut's theorem) - in this case $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$. We will often switch between Leibniz' and Newton's notations :

$$u_x = \frac{\partial u}{\partial x}$$

$$u_{xy} = (u_x)_y = \frac{\partial^2 u}{\partial y \partial x} \quad \text{etc.}$$

§3. <u>Complex differentiability</u> - The point of departure comes in the definition of $\mathbb{C}$-differentiability. Again, let $\Omega \subset \mathbb{C}$ be an open and connected subset and let $f:\Omega \to \mathbb{C}$ be a $\mathbb{C}$-valued function.

We say f is <u>$\mathbb{C}$-differentiable</u> at $z_0 \in \Omega$ if

$$\boxed{\lim_{\substack{h \to 0 \\ (h \in \mathbb{C})}} \frac{f(z_0+h) - f(z_0)}{h} \quad \text{exists}}$$

(denote this limit by $f'(z_0)$).

$f : \Omega \to \mathbb{C}$ is <u>$\mathbb{C}$-differentiable</u> means it is $\mathbb{C}$-diff. at every point $z_0 \in \Omega$. As usual, $\mathbb{C}$-differentiable implies continuous.

<u>Theorem.</u> – (Cauchy-Riemann equations)

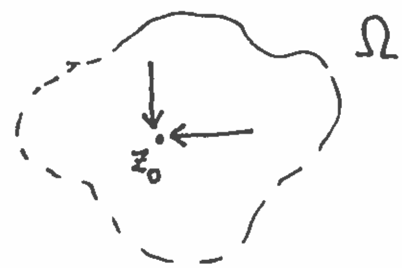
$f : \Omega \to \mathbb{C}$ $\quad f(x+iy) = u(x,y) + i\,v(x,y)$

is $\mathbb{C}$-differentiable if, and only if $\left(u, v \in C^1(\Omega)\right)$ and

$$\boxed{u_x = v_y \quad \& \quad u_y = -v_x}\,.$$

<u>Proof.</u> – Assume f is $\mathbb{C}$-differentiable, and let $z_0 = x_0 + iy_0 \in \Omega$.

We compute $f'(z_0)$ assuming $h = t \in \mathbb{R}$, and $h = it$ $(t \in \mathbb{R})$.



$$(1) \quad f'(z_0) = \lim_{\substack{t \to 0 \\ (t \in \mathbb{R})}} \frac{f(x_0 + t + iy_0) - f(x_0 + iy_0)}{t}$$

$$= u_x(x_0, y_0) + i\,v_x(x_0, y_0).$$

$$(2) \quad f'(z_0) = \lim_{t \to 0} \frac{f(x_0 + i(y_0 + t)) - f(x_0, y_0)}{it}$$

$$= -i\left(u_y(x_0, y_0) + i\,v_y(x_0, y_0)\right)$$

$$= v_y(x_0, y_0) - i\,u_y(x_0, y_0).$$

Cauchy-Riemann equations follow from comparing these two answers.

For the converse, we use Taylor's theorem for functions of two variables.

Assume $u, v \in C^1(\Omega)$ and let $z_0 = x_0 + iy_0 \in \Omega$
$h = a + ib$.

Then,
$$\frac{f(z_0+h) - f(z_0)}{h} - u_x(x_0,y_0) - i\,v_x(x_0,y_0)$$
$$= \frac{1}{h}\Big(u(x_0+a, y_0+b) + i\,v(x_0+a, y_0+b) - u(x_0,y_0) - i\,v(x_0,y_0) - a\,u_x(x_0,y_0) + b\,\underbrace{v_x(x_0,y_0)}_{-u_y} - i\,b\,\underbrace{u_x(x_0,y_0)}_{v_y} - i\,a\,v_x(x_0,y_0)\Big)$$
$$= \frac{1}{h}\Big(u(x_0+a, y_0+b) - u(x_0,y_0) - a\,u_x(x_0,y_0) - b\,u_y(x_0,y_0) + i\big(v(x_0+a, y_0+b) - v(x_0,y_0) - a\,v_x(x_0,y_0) - b\,v_y(x_0,y_0)\big)\Big)$$
$\to 0$ as $|h| = \sqrt{a^2+b^2} \to 0$ by Taylor's theorem. $\square$

Remarks.- (i) We only proved f is $\mathbb{C}$-diff. $\Rightarrow$ $u_x = v_y$ & $u_y = -v_x$

$u, v \in C^1(\Omega)$, $u_x = v_y$ & $u_y = -v_x$ $\Rightarrow$ f is $\mathbb{C}$-diff.

We will see next time that the hypothesis $u, v \in C^1(\Omega)$ is not needed. This is a consequence of Goursat's proof of Cauchy's theorem.

(ii) If $f = u + iv$ is $\mathbb{C}$-diff. (and $u, v \in C^2(\Omega)$ ← not needed by Cauchy-Goursat theorem)

then $\boxed{\begin{aligned} u_{xx} + u_{yy} &= 0 \\ v_{xx} + v_{yy} &= 0 \end{aligned}}$. Such functions are called <u>harmonic</u>.

So, f is $\mathbb{C}$-diff. $\Rightarrow$ $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are harmonic.

§4. Another form of Cauchy-Riemann equations. –

Using $x = \dfrac{z+\bar{z}}{2}$ (change of variables $(x,y) \leftrightarrow (z,\bar{z})$)

$y = \dfrac{z-\bar{z}}{2i}$ we get :

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y).$$

Thus C-R equations are equivalent to $\dfrac{\partial f}{\partial \bar{z}} = 0$, i.e., f does not depend on $\bar{z}$, only z. So, it is genuinely a function of one variable and we are allowed to use $\dfrac{df}{dz}$ instead of $\dfrac{\partial f}{\partial z}$.

e.g. polynomials and rational functions of z are $\mathbb{C}$-diff.

$z \mapsto |z|^2 = z \cdot \bar{z}$ is not $\mathbb{C}$-diff. Neither are $\mathrm{Re}(z)$, $\mathrm{Im}(z)$.

§5. Historical remarks. – Arithmetic and geometry of complex plane.

(a) Complex numbers were first recognized in the sixteenth century by Italian mathematicians in relation with the solution to the cubic equation

$$x^3 = 3px + 2q \Rightarrow x = \left(q + \sqrt{q^2 - p^3}\right)^{\frac{1}{3}} + \left(q - \sqrt{q^2 - p^3}\right)^{\frac{1}{3}}$$

Scipione del Ferro (1465 - 1526)

Girolamo Cardona (1501 - 1576) Niccolò Tartaglia (1499 - 1557)

Unlike the case of quadratic equations, where people simply disregard the case when discriminant < 0 - as no solutions., in the case of cubics there is always at least one real solution.

e.g. $x^3 = 15x + 4$ $(p = 5, q = 2)$ $\rightsquigarrow$ $\left(2 + \sqrt{-121}\right)^{1/3} + \left(2 - \sqrt{-121}\right)^{1/3}$

$x = 4$ solves this equation.

While Cardano freely manipulated these "impossible quantities", he called them "as subtle as they are useless".

(b) Rafael Bombelli (1526 - 1572) gave the first coherent treatment of arithmetic with complex numbers. He also "resolved the paradox" above by computing $\left(2 \pm \sqrt{-1}\right)^3 = 2 \pm \sqrt{-121}$.

(c) The term "imaginary" to describe $\sqrt{-1}$ is due to René Descartes (1596 - 1650), around 1620.

(d) The term "complex numbers" is due to Gauss (1777 - 1855), around 1822. He also used i = iota for $\sqrt{-1}$.

(e) Casper Wessel (1745 - 1818) around 1787; and Jean Argand (1768 - 1822), around 1806 - gave the geometric meaning to the algebraic operations in $\mathbb{C}$.

§6. Historical Remarks. – Cauchy · Riemann equations.

Two functions $u(x,y)$, $v(x,y)$ of two real variables are often viewed as a vector field on $\mathbb{R}^2$.

(a) Alexis Clairaut (1713 - 1765), in 1740, observed that if $\langle u,v\rangle$ represents force acting on a particle, then this vector field is conservative implies $u_y = v_x$.

(b) Jean d'Alembert (1717 - 1743), in 1752, imagined a 2-dim'l fluid flow with velocity $\langle P, Q\rangle$ and showed:

Flow is irrotational $\Leftrightarrow$ $P_y = Q_x$ (Zero circulation/curl)

Flow is incompressible $\Leftrightarrow$ $P_x = -Q_y$ (Zero flux)

He also observed that if both these equations hold - then $Q + iP$ is a function of one variable $z = x + iy$.

(c) Inspired by Maxwell's (1831- 1879) treatise on electromagnetism, around 1860, Felix Klein (1849 - 1925) wrote a monograph "On Riemann's theory of algebraic functions". Klein views $u(x,y)$ as a velocity potential – i.e., we have a flow with velocity vector $\langle u_x, u_y\rangle$. He showed

Laplace equation $u_{xx} + u_{yy} = 0$ $\leftrightarrow$ Flow is steady; and

Level curves $u = C_1$, $v = C_2$ always meet at right angles.