Lecture 1

§0. Recall - Lecture 0 : Let $\Omega \subset \mathbb{C}$ be an open, connected set and $f : \Omega \to \mathbb{C}$ a $\mathbb{C}$-valued function defined on Ω.

$$f(x+iy) = u(x,y) + i\,v(x,y).$$

Notation : $C^k(\Omega;\mathbb{R}) = \{ g : \Omega \to \mathbb{R} \mid$ partial derivatives of g, up to order k exist and are continuous $\}$.
$(k \in \mathbb{Z}_{\geq 0})$

$f \in C^k(\Omega;\mathbb{C})$ means $u, v \in C^k(\Omega;\mathbb{R})$.

Theorem. Assume $f \in C^1(\Omega;\mathbb{C})$. Then f is $\mathbb{C}$-differentiable if, and only if Cauchy-Riemann* equations hold

$$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$$ – Cauchy-Riemann equations.

Remark. - The hypothesis $f \in C^1(\Omega;\mathbb{C})$ is in fact superfluous. However, until we have proven Cauchy's theorem, I will keep writing it.

§1. Harmonic functions and some corollaries of Cauchy-Riemann equations.

Definition. - Let $g \in C^2(\Omega;\mathbb{R})$. We say g is harmonic if it satisfies Laplace* equation $\boxed{g_{xx} + g_{yy} = 0}$

Reading suggestion. - Find out why harmonic functions are called harmonic.

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* Luis-Augustin Cauchy (1789-1857)

Georg Friedrich Bernhard Riemann (1826-1866)

Pierre-Simon Laplace (1749-1827)

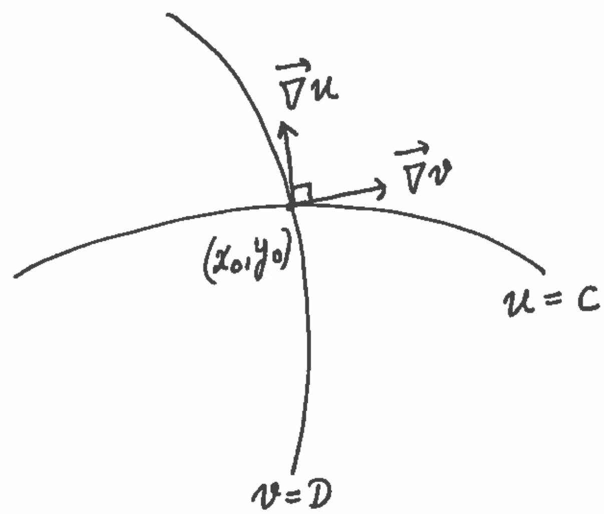
Proposition.- Assume $f: \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable and $f \in C^2(\Omega; \mathbb{C})$. Again $u = Re(f)$ and $v = Im(f)$.

(1) Both u and v are harmonic.

(2) Let $z_0 = x_0 + iy_0$ be a point where the level curves
$u(x,y) = C = u(x_0,y_0)$
$v(x,y) = D = v(x_0,y_0)$
meet. Then they meet at right angle, assuming $f'(z_0) \neq 0$.

Proof.- (1) $u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$ by Clairaut's theorem.

Similarly for v.



(2) $\vec{\nabla} u = \langle u_x, u_y \rangle$

$\vec{\nabla} v = \langle v_x, v_y \rangle$

are non-zero at (x_0, y_0); and

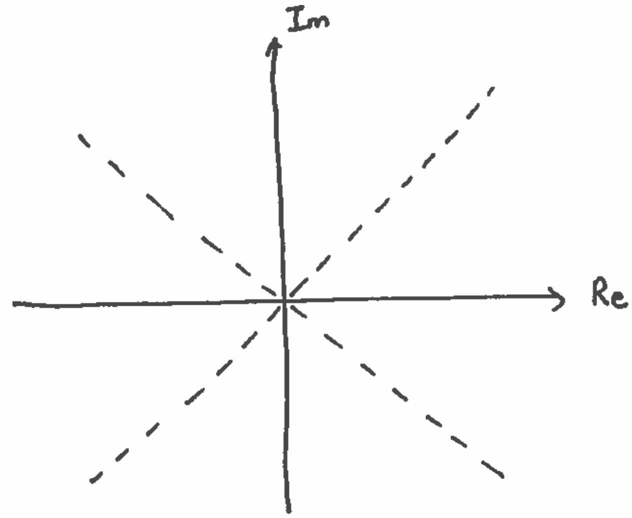
$$\vec{\nabla} u \cdot \vec{\nabla} v = u_x v_x + u_y v_y$$

$$= u_x(-u_y) + u_y u_x = 0. \qquad \square$$

Remarks.- (i) If $f'(z_0) = 0$, then the level curves are "reducible" - i.e., have more than one component.

e.g. $f(z) = z^2 = \underbrace{x^2 - y^2}_{u(x,y)} + \underbrace{(2xy)}_{v(x,y)} i \qquad f'(0) = 0.$

Dotted lines
$= \{x=y\} \cup \{x=-y\}$
= level curves
$u(x,y) = 0$.



Solid lines
= Real & Im. axes
= level curve $v(x,y)=0$

(ii) The orthogonality relation obtained above is the basis of the method of "steepest descent".

(iii) Given a harmonic function $u(x,y)$, we can consider the Cauchy-Riemann equations as PDE's (partial differential equations) defining another (harmonic) function $v(x,y)$

$$\frac{\partial v}{\partial x} = -u_y$$

$$\frac{\partial v}{\partial y} = u_x$$

so that $u + iv$ is $\mathbb{C}$-differentiable.

The consistency (or integrability, or cross-derivative) condition for these PDE's = Laplace equation for u

We can always solve such a system <u>locally</u> (i.e. in an open neighbourhood of a point) - but <u>not globally</u> - if Ω is not <u>simply-connected</u> (for later).

Such a solution $v(x,y)$ is called a harmonic conjugate of $u(x,y)$. It is unique up to a real constant.

§2. A rough sketch of this course –

$\boxed{\text{Object of interest : } f : \Omega \to \mathbb{C} \quad \mathbb{C}\text{-diff / holomorphic / analytic}}$

Geometric side

- Mapping properties of hol. fns.
- Riemann mapping theorem
- Riemann Surfaces and uniformization.

Functional side

- Differential equations
- Difference equations
- Integral equations

- Boundary Value Problems
- Potential Theory

- Asymptotic analysis
- Resummation methods
  Integral transforms

(This is NOT a Venn diagram - lots of intersections)

Foundational Results :

Uniform Convergence. { Taylor / Laurent series and identity theorem
Weierstrass' and Abel's Theorems

Cauchy's integral formula

Cauchy-Riemann Equations

§3. Exponential and logarithm. (Leonhard Euler 1707 - 1783)

Examples of $\mathbb{C}$-diff. fns so far - Polynomials and rational functions.

Next - e^z and $\log(z)$.

Historical interlude. - $e = 2.71828\ldots$ (Euler's constant)

- Jacob Bernoulli (1655-1705) was the first to come across this number, in the context of a problem of compound interest. He observed that the following limit exists

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71\ldots$$

Check: to get accuracy of 4 decimal places $n \approx 10^5$.

Interestingly, in some communications between Leibniz & Huygens, this limit is denoted by b for Bernoulli.

- Euler came across the same number while studying a problem of "population growth".

$$f'(t) = k\cdot f(t) \quad \text{(rate of growth is proportional to population itself)}$$

$$f(0) = C \quad \text{(initial population)}$$

→ Let $K = C = 1$ for convenience and solve it formally to get

$$f(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

$$f(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \approx 2.7182\ldots$$

Check: just need 7 terms to get accuracy up to 4 decimal places.

Euler used the letter 'e' (possibly to indicate <u>ex</u>plosion) for this constant and later proved that $e = b$.

The general case $\begin{array}{l} f'(t) = K f(t) \\ f(0) = C \end{array}$ is solved by $f(t) = C \cdot e^{Kt}$.

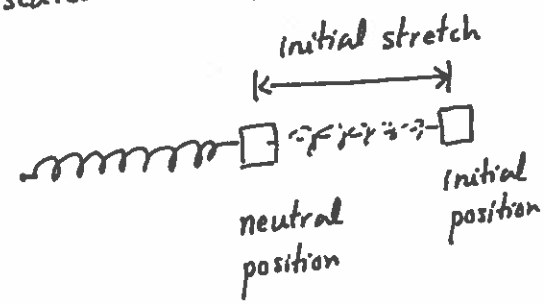
- The oscillatory equation (or differential equation for a spring):

$$f''(t) = -K f(t) \quad (K>0) \text{ - spring constant}$$

$$f(0) = 1 \quad \text{(initial stretching - I scaled it to 1)}$$

(rest at $t=0$) $f'(0) = 0$

is solved by $\cos(\sqrt{K}\, t)$



$f''(t) = -f(t)$ has general solution $A\cos(t) + B\sin(t)$

Euler ~~also~~ observed that e^{it} also solves this equation; with initial conditions $e^{it}\big|_{t=0} = 1$; $(e^{it})'\big|_{t=0} = (i \cdot e^{it})\big|_{t=0} = i$.

Hence obtaining
"The most remarkable formula in mathematics"
- Feynman (Lectures on physics, vol. I, Ch. 22)

$$\boxed{e^{it} = \cos(t) + i\sin(t)}$$

Ex. Prove, using binomial theorem, that for any two commuting variables A, B; $e^{A+B} = e^{A} \cdot e^{B}$ $\left(e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \right)$.

Thus $e^z = e^x \cdot e^{iy} = e^x(\cos(y) + i\sin(y))$.

Verify Cauchy-Riemann equations for $u(x,y) = e^x\cos(y)$
$v(x,y) = e^x\sin(y)$.

Sol.: $u_x = e^x\cos(y)$ $\quad v_y = e^x\cos(y)$

$u_y = -e^x\sin(y)$ $\quad v_x = e^x\sin(y)$ $\quad\square$

$$\frac{d}{dz}(e^z) = u_x + i v_x = e^z.$$

§4. Properties of e^z:

$$\left.\begin{aligned} e^0 &= 1 \\ \frac{d}{dz}e^z &= e^z \end{aligned}\right\} \leftarrow \text{defines } e^z \text{ uniquely.}$$

$$e^{z_1+z_2} = e^{z_1}\cdot e^{z_2}. \qquad e^z = 1 \iff z \in 2\pi i\mathbb{Z}.$$

$$e^{z+2\pi i n} = e^z \quad (\text{1-periodic of period } 2\pi i)$$

$$\mathrm{Re}(e^z) = e^x\cos(y), \quad \mathrm{Im}(e^z) = e^x\sin(y)$$

$$|e^z| = e^{\mathrm{Re}(z)}, \quad \arg(e^z) = \mathrm{Im}(z)$$

§5. Geometric mapping under $z \mapsto e^z$

To "visualize" functions of a complex variable, people usually sketch some lines / regions in $\mathbb{C}$-plane (where z lives) and their image under the function — again in $\mathbb{C}$-plane (where $w = f(z)$ lives)

z-plane $\dashrightarrow \boxed{w = e^z} \dashrightarrow$ w-plane