Lecture 2

Contour integrals and Cauchy's theorem.

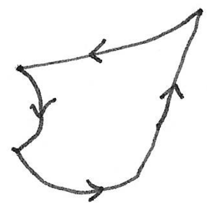
§1. Paths, or parametric curves in $\mathbb{C}$. – By a (parametrized) path, we mean a continuous function $\gamma : [a,b] \to \mathbb{C}$ $\quad (a, b \in \mathbb{R},\ a<b)$.

We will always assume that our paths are piecewise C^1 – i.e.

$$[a,b] = [a,t_1] \cup [t_1,t_2] \cup \dots \cup [t_{n-1},t_n] \cup [t_n,b] \quad \text{s.t.}$$

$$\gamma'(t) \text{ exists and is continuous for } t \notin \{a, t_1, \dots, t_n, b\}.$$

Simple = no self crossings, except possibly at end points.



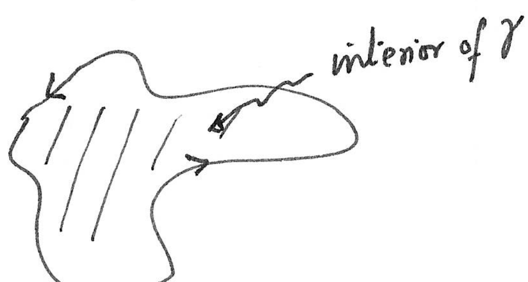
example of a piecewise C^1 (simple, closed) path.

Closed = $\gamma(a) = \gamma(b)$

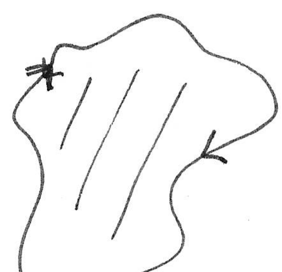
Contour = simple, closed, counterclockwise oriented path

Orientation: if γ is a simple, closed path, then $\mathbb{C} \setminus \operatorname{image}(\gamma)$ has 2 connected components (Jordan curve theorem).

We say γ is positively oriented (or counterclockwise) if the bounded component of $\mathbb{C} \setminus \operatorname{image}(\gamma)$ is always to the left of γ



γ (positively oriented)



negatively oriented.

Shaded region = bounded component of $\mathbb{C} \setminus \operatorname{image}(\gamma)$

§2. Line integral Let $\Omega \subset \mathbb{C}$ be an open & connected subset

$f : \Omega \to \mathbb{C}$ a continuous $\mathbb{C}$-valued function

$\gamma : [0,1] \to \Omega$ a piecewise C^1 path. [I took $a=0$, $b=1$ for

$$\int_\gamma f(z)\,dz := \int_0^1 f(\gamma(t))\,\gamma'(t)\,dt \qquad \text{if } \gamma \text{ is } C^1$$

$$= \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} f(\gamma(t))\cdot\gamma'(t)\,dt \qquad \text{if } t_0 = a < t_1 < \dots < t_n < t_{n+1} = 1$$

are "corner" points.

Recall - length of γ is defined as

$$L(\gamma) = \int_0^1 |\gamma'(t)|\,dt.$$

Properties. - (i) Linearity : $\displaystyle\int_\gamma (\alpha f(z) + \beta g(z))\,dz = \alpha \int_\gamma f(z)\,dz + \beta \int_\gamma g(z)\,dz.$

(ii) If $\gamma_1 : [a,b] \to \Omega$ and $\gamma_2 : [b,c] \to \Omega$ are such that $\gamma_1(b) = \gamma_2(b)$, one can concatenate $\gamma_1 \cdot \gamma_2 : [a,c] \to \Omega$.

$$\int_{\gamma_1\cdot\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

(iii) $\gamma : [a,b] \to \Omega \quad \underset{\text{(reverse)}}{\rightsquigarrow} \quad \gamma^{-1} : [-b,-a] \to \Omega$

$$\gamma^{-1}(-t) = \gamma(t)$$

$$\int_{\gamma^{-1}} f(z)\,dz = -\int_{\gamma} f(z)\,dz.$$

(iv) Independence from parametrization.

$\gamma : [a,b] \to \Omega \qquad \tau : [c,d] \to [a,b]$ piecewise C^1 bijection

$\rightsquigarrow \mu : [c,d] \to \Omega$

$$\mu(t) := \gamma(\tau(t))$$

$$\Rightarrow \quad \int_{\gamma} f(z)\,dz = \int_{\mu} f(z)\,dz.$$

(v) Triangle inequality. Let $M = \mathrm{Max}\{|f(z)| : z \in \mathrm{image}(\gamma)\}$

L = length of γ.

Then
$$\boxed{\left| \int_{\gamma} f(z)\,dz \right| \leq M \cdot L}$$

(vi) (Fundamental theorem of calculus) If $\exists\, F : \Omega \to \mathbb{C}$ $\mathbb{C}$-diff. (primitive, or antiderivative of f).

s.t. $F'(z) = f(z)$, then

$$\int_{\gamma} f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

§3. Morera's theorem. Again $\Omega \subset \mathbb{C}$ is open and connected, and $f : \Omega \to \mathbb{C}$ is a continuous function. We say $F : \Omega \to \mathbb{C}$ is a primitive (or antiderivative) of f if F is $\mathbb{C}$-differentiable and $F'(z) = f(z)$ $\forall z \in \Omega$.

Theorem. – The following are equivalent.

(a) f admits a primitive

(b) For any piecewise C^1 $\gamma : [a,b] \to \Omega$,

$$\int_\gamma f(z)\,dz \text{ depends only on } \gamma(a) \;\&\; \gamma(b)$$

(c) Same statement as (b) - for γ consisting entirely of horizontal and vertical line segments (a zig-zag path).

Fact (topological - using Heine-Borel theorem - closed & bounded $\Rightarrow$ compact, i.e., every open cover has a finite subcover)

if $\Omega \subset \mathbb{C}$ is open & (path) connected, then any two points in Ω can be joined by a zig-zag path.

Proof. – (a) $\Rightarrow$ (b) by (vi) of §2 above.

(b) $\Rightarrow$ (c) obviously. Now we prove (c) $\Rightarrow$ (a).

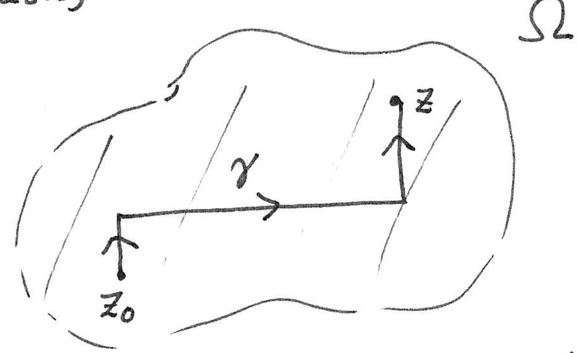
Giacinto Morera (1856–1907)

Fix $z_0 \in \Omega$ and define $F : \Omega \to \mathbb{C}$ by

$$F(z) := \int_\gamma f(w)\,dw$$

where γ is <u>any</u> zig-zag path joining z_0 to z.

(well-defined by our hypothesis)



<u>Claim</u>.- $F : \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable and $F'(z) = f(z)\ \forall z \in \mathbb{C}$.

Let us write
$$f = u + iv$$
$$F = U + iV.$$

Choice of γ - immaterial by assumption (c).

It is enough to show that
$$U_x = V_y = u$$
$$U_y = -V_x = -v$$

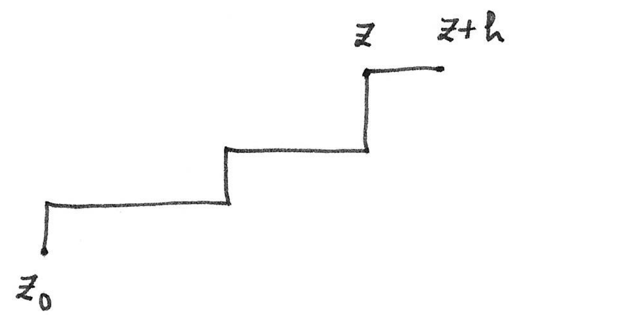
($u, v \in C^0(\Omega;\mathbb{R})$ - so it shows $U, V \in C^1(\Omega;\mathbb{R})$ & C-R eq^{n}s for U, V)

Assume $h \in \mathbb{R}$ is small enough so that $z + h \in \Omega$.

$$F(z+h) - F(z) = \int_{\gamma\cdot\gamma_H} f(z')\,dz' - \int_\gamma f(z')\,dz'$$

$$= \int_{\gamma_H} f(z')\,dz'$$

$$= \int_0^1 f(z+th)\cdot h \cdot dt$$



γ : zig-zag joining z_0 to z

$\gamma_H(t) = z + th$ joins z to $z+h$ $(0 \le t \le 1)$

$$\Rightarrow \quad \frac{F(z+h)-F(z)}{h} = \int_0^1 f(z+th)\,dt .$$

Easy exercise: $\displaystyle\lim_{h\to 0} \int_0^1 f(z+th)\,dt = f(z)$.

Hence, $U_x + i V_x = u + iv$. Same calculation with γ_{vertical} instead of horizontal shows $V_y - i U_y = u + iv$. Hence, $F \in C^1(\Omega;\mathbb{C})$ (as u and v are assumed to be continuous) and C-R equations hold for F, implying that F is $\mathbb{C}$-differentiable and

$$F'(z) = f(z) \quad \forall\, z \in \Omega. \qquad \square$$

§4. <u>Cauchy's theorem.</u> – Again let $\Omega \subset \mathbb{C}$ be open and connected.

We say Ω is <u>simply-connected</u> if for every simple, closed path in Ω, $\gamma : [0,1] \to \Omega$, the interior$(\gamma) \subset \Omega$.

e.g. $\mathbb{C}$, $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, $\mathbb{D}$ = unit disc $= \{z : |z| < 1\}$ are simply-connected.

$\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, $\mathbb{D}^\times = \mathbb{D} \setminus \{0\}$ are not simply-connected.

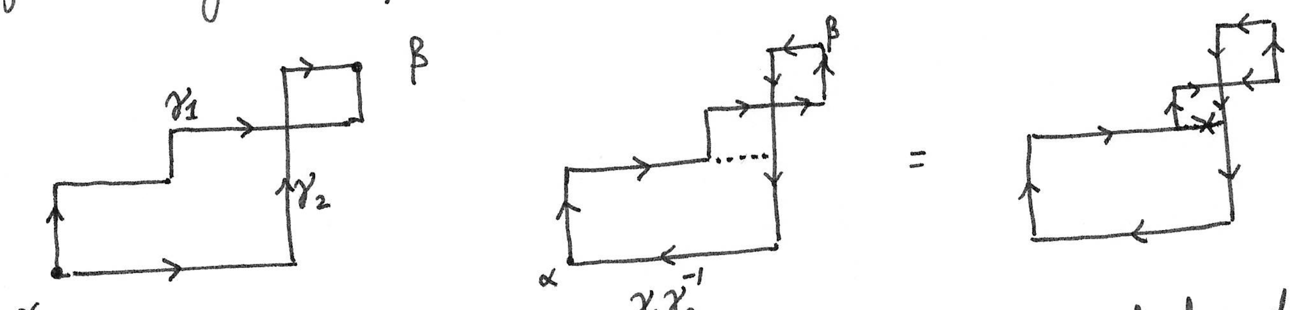
<u>Theorem.</u> – Assume
- Ω is simply-connected.
- $f : \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable
- $\gamma : [0,1] \to \Omega$ is <u>any</u> closed path (loop) (piecewise C^1 – always)

Then
$$\boxed{\int_\gamma f(z)\,dz = 0.}$$

Proof. Using Morera's theorem (§3 above) it is enough to show that $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$ for any two zig-zag paths with same end points. As

$$\int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz = \int_{\gamma_1\gamma_2^{-1}} f(z)\,dz,$$

this is equivalent to proving that $\int_{\gamma} f(z) = 0$ for any closed zig-zag path. Any such path is concatenation of boundries of rectangles.



γ_1, γ_2 : two zig-zag paths from α to β

closed zig zag path based at α

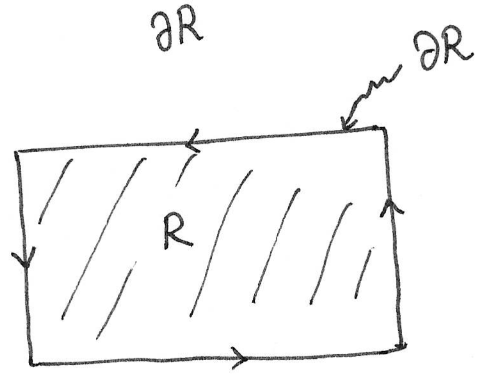
Sum of boundaries of rectangles.

Hence, Cauchy's theorem follows from its special case when γ is boundary of a rectangle

To prove: Let R be a rectangle in Ω. Then $\int_{\partial R} f(z)\,dz = 0$.

Let D = length of the diagonal of R.

P = perimeter of R.

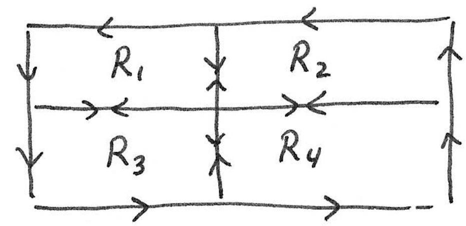


Claim. - Given any $\varepsilon > 0$, $\left| \int_{\partial R} f(z)\,dz \right| < \varepsilon \cdot D \cdot P.$

(it is clear that this claim implies $\int_{\partial R} f(z)\,dz = 0$).

Proof of the claim. - Divide R into 4 pieces, so that

$$\int_{\partial R} f(z)\,dz = \sum_{j=1}^{4} \int_{\partial R_j} f(z)\,dz.$$



Choose R_k $(k = 1,2,3 \text{ or } 4)$ so that $\left| \int_{\partial R_k} f(z)\,dz \right| \geq \left| \int_{\partial R_j} f(z)\,dz \right|$ for every $j \in \{1,2,3,4\}$.

and call it $R^{(1)}$.

By triangle inequality, $\left| \int_{\partial R} f(z)\,dz \right| \leq 4 \left| \int_{\partial R^{(1)}} f(z)\,dz \right|$

Now repeat these steps for $R^{(1)}$ to get $R^{(2)}$, and so on.

We get a descending chain of closed (shrinking) rectangles

$$R^{(0)} = R \supset R^{(1)} \supset R^{(2)} \supset \cdots$$

If D_n = diameter of $R^{(n)}$, then $D^{(n)} = \frac{1}{2^n} D$

P_n = perimeter of $R^{(n)}$ $\quad P^{(n)} = \frac{1}{2^n} P$

and $\left| \int_{\partial R} f(z)\,dz \right| \leq 4^n \left| \int_{\partial R^{(n)}} f(z)\,dz \right|.$

Now $\displaystyle\bigcap_{n=1}^{\infty} R^{(n)} = \{z_0\}$ (this is almost an axiom - can be shown to be equivalent to completeness axiom of $\mathbb{R}$).

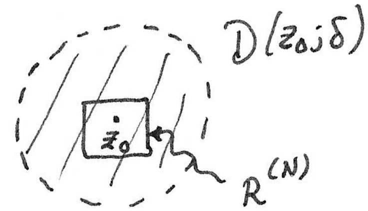
Let $\varepsilon > 0$ be as in the claim and choose $\delta > 0$ such that

$$|z - z_0| < \delta \quad \Rightarrow \quad \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

(by $\mathbb{C}$-diff. of f - such $\delta > 0$ exists.)

Let $N > 0$ be large enough so that $R^{(N)} \subset D(z_0;\delta) = \{z \mid |z - z_0| < \delta\}$

As constant fn. and z have antiderivatives,



$$\int_{\partial R^{(N)}} f(z_0)\, dz = 0 = \int_{\partial R^{(N)}} (z - z_0) f'(z_0)\, dz .$$

Hence we get $$\int_{\partial R^{(N)}} f(z)\, dz = \int_{\partial R^{(N)}} \left(f(z) - f(z_0) - (z - z_0) f'(z_0) \right) dz$$

$$\Rightarrow \left| \int_{\partial R^{(N)}} f(z)\, dz \right| < \int_{\partial R^{(N)}} \varepsilon \cdot |z - z_0| \cdot |dz| \le \varepsilon \cdot \underset{z \in \partial R^{(N)}}{\mathrm{Max}} (|z - z_0|) \cdot \underbrace{\mathrm{Length}(\partial R^{(N)})}_{\text{perimeter of } R^{(N)}}$$

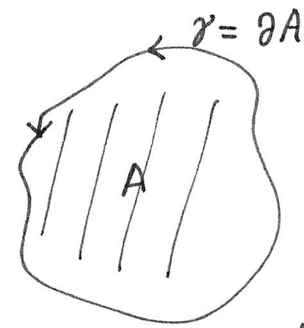
$$\le \varepsilon \cdot D_N \cdot P_N = \frac{\varepsilon \cdot DP}{4^N} .$$

As $\displaystyle \left| \int_{\partial R} f(z)\, dz \right| \le 4^N \left| \int_{\partial R^{(N)}} f(z)\, dz \right|$, our claim follows.

$\square$

§5. Historical remarks. - (1) Augustin-Louis Cauchy (1789-1857) first published a proof of Theorem §4 above, under an additional hypothesis that u_x, u_y, v_x, v_y are continuous, in 1841. His proof was based on Green*'s theorem (a special case of Stokes'* theorem)

Green's theorem (1828) Let $P, Q \in C^1(\Omega)$ and $\rightsquigarrow$



in Ω

$$\int_{\partial A} P\,dx + Q\,dy = \iint_A (Q_x - P_y)\,dx\,dy$$

γ a contour bounding a closed set $A \subset \Omega$.

Cauchy's original proof

$$\mathrm{Re} \int_\gamma f(z)\,dz = \int_\gamma u\,dx - v\,dy = \iint_A (-v_x - u_y)\,dx\,dy = 0 \text{ by C-R eq}^{\text{n}}\text{s.}$$

(2) The more general statement and its proof that we have presented in §3 are due to Eduard Goursat (1858-1936).

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* Georg Green 1793-1841. George Gabriel Stokes 1819-1903