Lecture 3

§0. Recall that we have proved the following results so far.

$$\boxed{\bullet\ \underset{\text{open, connected}}{\Omega \subset \mathbb{C}} \qquad \bullet\ f : \Omega \to \mathbb{C} \text{ continuous.}}$$ ← notation fixed throughout this Lecture.

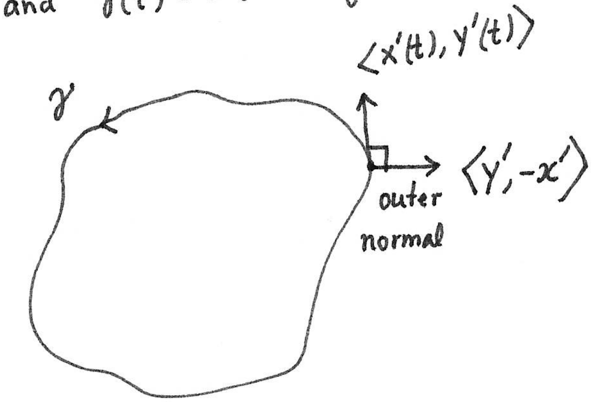
(1) Assuming $f \in C^1(\Omega;\mathbb{C})$, f is $\mathbb{C}$-diff. $\Leftrightarrow$ Cauchy-Riemann eq^{n}'s hold. (Lecture 0, §3)

(2) TFAE : (a) f admits a primitive (b) $\int_\gamma f(z)\,dz$ only depends on γ: for every piecewise C^1 $\gamma : [0,1] \to \Omega$ (c) Same as (b) for any zig-zag γ.
(Morera's theorem, Lecture 2, §3)

(3) $\begin{pmatrix} f : \mathbb{C}\text{-diff.} \\ \Omega : \text{simply. connected} \end{pmatrix}$ $\int_\gamma f(z)\,dz = 0$ for any closed γ (i.e., a loop).

§1. Physical significance of contour integrals. (cf Lecture 0, §6 (b)).

Let us write $f(x+iy) = u(x,y) + i\,v(x,y)$ and $\gamma(t) = x(t) + i\,y(t)$.



By our definition,

$$\int_\gamma f(z)\,dz = \int_0^1 \Big(u(x(t),y(t)) + i\,v(x(t),y(t))\Big)\cdot (x'(t) + i\,y'(t))\,dt$$

So,

$$\operatorname{Re}\int_\gamma f(z)\,dz = \int_0^1 \Big(u(x(t),y(t))\,x'(t) - v(x(t),y(t))\,y'(t)\Big)\,dt$$

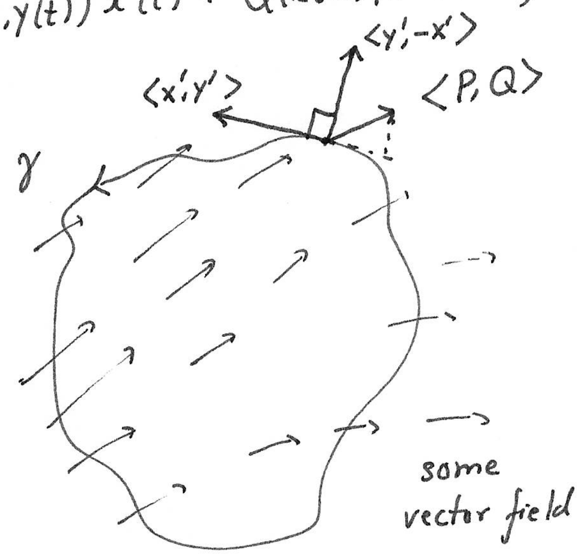
$$\operatorname{Im}\int_\gamma f(z)\,dz = \int_0^1 \Big(u(x(t),y(t))\,y'(t) + v(x(t),y(t))\,x'(t)\Big)\,dt$$

Given a (velocity) vector field $\vec{V} = \langle P(x,y),\, Q(x,y) \rangle$, the <u>flow along γ</u>, called <u>circulation</u>, is defined as

$$\text{Circulation}_\gamma(\vec{V}) = \int \vec{V} \cdot \vec{\gamma}'(t)\, dt$$

$$= \int_0^1 \Big(P(x(t),y(t))\, x'(t) + Q(x(t),y(t))\, y'(t) \Big)\, dt$$

($\gamma = \langle x(t), y(t) \rangle$ a parametric, closed curve in $\mathbb{R}^2$).



Similarly <u>flow across γ</u>, or <u>flux</u> is defined as

$$\text{Flux}_\gamma(\vec{V}) = \int P\,dy - Q\,dx .$$

Component of $\langle P, Q \rangle$ along γ
$= \langle P, Q \rangle \cdot \langle x', y' \rangle$

Component of $\langle P, Q \rangle$ across γ
$= \langle P, Q \rangle \cdot \langle y', -x' \rangle$

Comparing with formulae on the previous page, we get

$$\boxed{\int_\gamma f(z)\, dz = \text{Circulation}_\gamma(\langle u, -v \rangle) + i\, \text{Flux}_\gamma(\langle u, -v \rangle)}$$

In these "physical terms" Cauchy's theorem seem to be "known" to d'Alembert around 1752.

§2. Application 1 of Cauchy's theorem.-

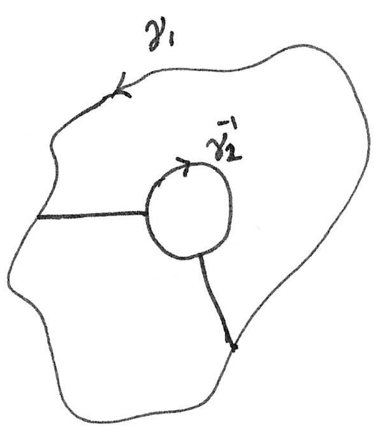
Principle of contour deformation.- Again $\Omega \subset \mathbb{C}$ is open and connected, $f : \Omega \to \mathbb{C}$ is $\mathbb{C}$-diff.

Assume γ_1 is a contour in Ω and γ_2 is a contour in interior(γ_1) such that interior$(\gamma_1) \cap$ exterior$(\gamma_2) \subset \Omega$.

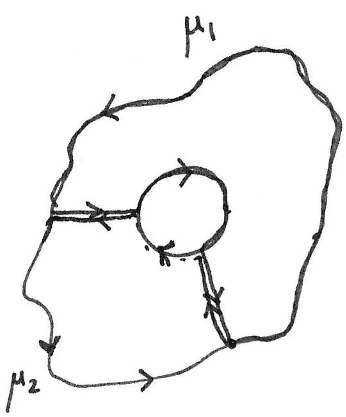
Then $$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

"Proof": - see this picture

add and subtract this and use Cauchy's theorem.



$$\int_{\gamma_1} - \int_{\gamma_2} =$$



$$\int_{\mu_1} + \int_{\mu_2} = 0$$ by Cauchy's theorem.

$\square$

§3. Important example. Let $n \in \mathbb{Z}$ and take γ any contour in $\Omega = \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Then

$$\int_\gamma z^n \, dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1. \end{cases}$$

Proof. – By Contour deformation replace γ by $C(0;r)$ = counter clockwise circle of radius r, centered at 0. ($r \in \mathbb{R}_{>0}$).

(Note: if $n \neq -1$, $z^n = \frac{d}{dz}\left(\frac{z^{n+1}}{n+1}\right)$ has a primitive, so the integral is 0 by ~~Morera's theorem~~ fund. thm. of Calculus – see Lecture 2, §2 (vi)).

$$\int_\gamma \frac{dz}{z} = \int_{C(0;r)} \frac{dz}{z} = \int_0^{2\pi} \frac{r i e^{i\theta} d\theta}{r e^{i\theta}} = i \int_0^{2\pi} d\theta = 2\pi i. \qquad \square$$

$C(0;r) = \{re^{i\theta} : 0 \leq \theta \leq 2\pi\}$

Remark. – This proves that $\frac{1}{z}$ does not have a primitive on $\mathbb{C}^\times$ which would have been logarithm.

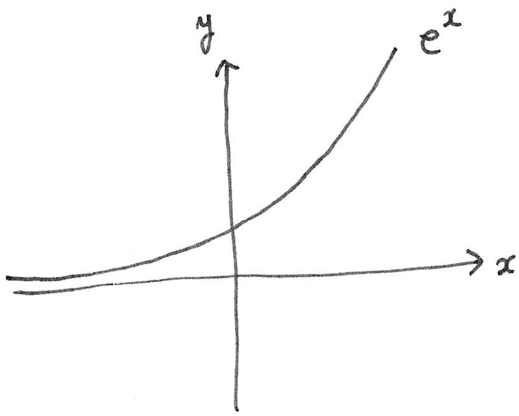
However – if we agree to be content with a simply-connected subset of $\mathbb{C}^\times$ – we can define logarithm as a primitive of $\frac{1}{z}$.

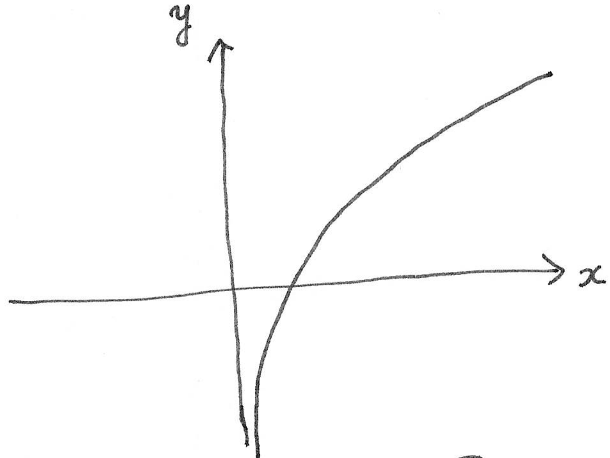
The so-called principal branch of logarithm is thus defined as

$$\boxed{\begin{aligned} \log &: \mathbb{C} \setminus \mathbb{R}_{<0} \to \mathbb{C} \\ \log(z) &:= \ln(|z|) + i \arg(z) \; ; \; \arg(z) \in (-\pi, \pi) \end{aligned}}$$

§4. Logarithm function. The function $\log_{10}(x)$ was originally defined by John Napier (1550 - 1617) and was heavily used in seventeenth century as a computational mechanism.

In mathematics we are now used to "natural logarithm" - i.e., logarithm in base 'e' (Euler's constant). Over $\mathbb{R}$ it is defined as the inverse of the exponential function.





$e^x : \mathbb{R} \longrightarrow \mathbb{R}_{>0}$ $\qquad$ $\ln(x) : \mathbb{R}_{>0} \to \mathbb{R}$

The main issue in extending this definition to $\mathbb{C}^\times$; is that e^z is no longer a one-to-one function.

$$e^0 = e^{\pm 2\pi i} = e^{\pm 4\pi i} = \cdots = 1 \quad ; \text{ so}$$

$$\text{"log"}(1) = 0 \text{ or } \pm 2\pi i, \ldots$$ which one should we take?

This is the same problem as a consistent choice of argument.

<u>Note</u> : $e^{\text{"log"}(z)} = z \Rightarrow \dfrac{d}{dz} e^{\text{"log"}(z)} = e^{\text{"log"}(z)} \cdot \dfrac{d}{dz}\text{"log"}(z) = 1$

$$\Rightarrow \frac{d}{dz}\text{"log"}(z) = \frac{1}{z}.$$

So, no matter how we define "log" - as a (local) inverse to e^z - it must solve the differential equation

$$\frac{d}{dz}\,\text{"log"}(z) = \frac{1}{z}.$$

Let us put the initial condition

$$\text{"log"}(1) = 0.$$

This defines "log"(z) only locally - §3 Example shows that there is no global solution to the problem.

$$w = e^{z} = e^{\mathrm{Re}(z)} \cdot e^{i(\cancel{\arg(z)}\ \mathrm{Im}(z) + 2\pi n)} \quad (n \in \mathbb{Z}) \;;\; |w| = e^{\mathrm{Re}(z)}$$

$$\Rightarrow z = \text{"log"}(w) = \ln|w| + i(\arg(w) + 2\pi n)$$

§5. Conventions. - By the "principal branch" of logarithm, we mean $\boxed{\log : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to \mathbb{C}}$

$$\log(z) = \ln|z| + i\arg(z) \;;\; \arg(z) \in (-\pi, \pi).$$

One must keep in mind that this is one of many choices.

Some authors use $\mathrm{Log} : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to$ "Subsets of $\mathbb{C}$"

$$\mathrm{Log}(z) = \ln|z| + i\arg(z) + 2\pi i\,\mathbb{Z}$$

as the totality of all the choices - and call it a "multi-valued" function.

Remark.- Unlike $\ln(x)$ $(x \in \mathbb{R}_{>0})$,

$$\log(z_1 z_2) \neq \log(z_1) + \log(z_2).$$ First of all

$z_1, z_2 \in \mathbb{C}^\times$ ~~do~~ could give $z_1 z_2 \in \mathbb{R}_{<0}$

$z_1, z_2 \notin \mathbb{R}_{<0}$ ($\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ is not a multiplicative subgroup of $\mathbb{C}^\times$.)

(Johann Bernoulli (1667 - 1748) was stuck here - in an attempt to define $\log(z)$ - His "counterexample" was

$$0 = \log(1) = \log(-1)^2 = 2\log(-1) \; ; \text{ so } \log(-1) = 0.)$$

Euler was the first to realize that all one can say that $\log(z_1 z_2) - \log(z_1) - \log(z_2) \in 2\pi i \mathbb{Z}$. $\left(\begin{matrix} z_1, z_2, z_1 z_2 \\ \in \mathbb{C} \setminus \mathbb{R}_{\leq 0} \end{matrix}\right)$

Weierstrass' theory of analytic continuations and Riemann surfaces was motivated by the problem to make sense of "multi-valued" functions - we will return to this later.

§6. Power function - for any $\alpha \in \mathbb{C}$, define

$$z^\alpha := e^{\alpha \log(z)} : \underset{\underset{\text{choice}}{\uparrow}}{\mathbb{C} \setminus \mathbb{R}_{\leq 0}} \to \mathbb{C}.$$

Karl Theodor Wilhelm Weierstrass (1815 - 1897)

More precisely - z^α is the/a solution to

$$\frac{d}{dz} f(z) = \frac{\alpha}{z} \cdot f(z).$$

Initial Condition (say) $f(1) = 1$.

- if $\alpha \in \mathbb{Z}_{\geq 0}$, we have a global solution $\mathbb{C} \xrightarrow[(\alpha \in \mathbb{Z}_{\geq 0})]{z^\alpha} \mathbb{C}$
- if $\alpha \in \mathbb{Z}_{<0}$, we have a global solution $\mathbb{C}^\times \xrightarrow[(\alpha \in \mathbb{Z}_{<0})]{z^\alpha} \mathbb{C}$
- Otherwise, z^α is "multi-valued"