Lecture 4

§0. Recall that we proved Cauchy's theorem and principle of contour deformations.

$\Omega \subset \mathbb{C}$, open, connected and simply-connected and $f : \Omega \to \mathbb{C}$ $\mathbb{C}$-differentiable implies $\boxed{\int_C f(z)\,dz = 0}$ for any closed, piecewise C^1 curve $C \subset \Omega$.

$\Omega \subset \mathbb{C}$ open, connected, $f : \Omega \to \mathbb{C}$ $\mathbb{C}$-differentiable, and $\gamma_1, \gamma_2 : [0,1] \to \Omega$ contour s.t. γ_2 lies in interior (γ_1) $(\cap \Omega)$ and interior $(\gamma_1) \cap$ exterior $(\gamma_2) \subset \Omega$. Then

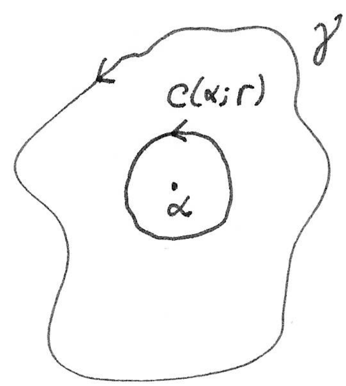
$$\boxed{\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz}$$

§1. Cauchy's integral formula,

Let $\Omega \subset \mathbb{C}$ be an open, connected set, $\gamma : [0,1] \to \Omega$ a contour in Ω s.t interior $(\gamma) \subset \Omega$. Let $\alpha \in$ Interior (γ).

Then

$$\boxed{f(\alpha) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-\alpha}\,dz}$$



Proof.- By principle of contour deformation

$$\int_\gamma \frac{f(z)}{z-\alpha}\,dz = \int_{C(\alpha;r)} \frac{f(z)}{z-\alpha}\,dz$$ for any $r>0$ small enough so that $\overline{D}(\alpha,r) = \{z : |z-\alpha| \leq r\}$ lies in $(\cancel{\Omega} \cap)\ \text{interior}(\gamma)$.

<u>Claim</u>: $$\int_{C(\alpha;r)\ (\text{or } \gamma)} \frac{f(z)-f(\alpha)}{z-\alpha}\,dz = 0.$$

<u>Proof.</u> As $\displaystyle\lim_{z\to\alpha} \frac{f(z)-f(\alpha)}{z-\alpha} = f'(\alpha)$ exists, given $\varepsilon>0$, we can find $\delta>0$ s.t.

$$0<|z-\alpha|<\delta \implies \left| f(z)-f(\alpha) - (z-\alpha) f'(\alpha)\right| < \varepsilon . |z-\alpha|.$$

So, for every $0<r<\delta$, we get

$$\left| \int_\gamma \left(\frac{f(z)-f(\alpha)}{z-\alpha} - f'(\alpha)\right) dz \right|$$

$$= \left| \int_{C_{a,r}} \left(\frac{f(z)-f(\alpha)}{(z-\alpha)} - f'(\alpha)\right) dz \right|$$

$$< \varepsilon . 2\pi r.$$ Letting $r\to 0$, we conclude

$$\int_\gamma \frac{f(z)-f(\alpha)}{z-\alpha}\,dz = \int_\gamma f'(\alpha)\,dz = 0.$$ The claim follows. $\square$

The claim implies $\displaystyle\int_\gamma \frac{f(z)}{z-\alpha}\,dz = \int_\gamma \frac{f(\alpha)}{z-\alpha}\,dz$

$$= \int_{C(a;r)} \frac{f(\alpha)}{z-\alpha}\,dz = f(\alpha)\int_{C(0;r)} \frac{dz}{z} = 2\pi i f(\alpha)$$

(principle of contour deformation) $\square$

§2. <u>Remarks.</u> Cauchy's integral formula generalizes without any difficulty to

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-\alpha)^{n+1}}\,dz$$

proving that $\mathbb{C}$-differentiable $\Rightarrow$ $\mathbb{C}$-differentiable to all orders.

Henceforth, we call $\mathbb{C}$-differentiable functions <u>holomorphic</u>.

After getting through the notion of <u>uniform convergence</u>, we will be able to prove

$$f(z) = \sum_{n=0}^{\infty} (z-\alpha)^n \left(\frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-\alpha)^{n+1}}\,ds \right)$$ Taylor series expansion near α

uniformly on compact subsets of $D(\alpha;R) = \{z : |z-\alpha| < R\}$

where $R =$ distance between α and $\partial\overline{\Omega}$, the boundary of $\overline{\Omega}$ (distance to the nearest singularity).

Hence, for $\mathbb{C}$-differentiability notion:

$$\begin{array}{ccccc} f'(\alpha) \text{ exists} & \Rightarrow & f^{(n)}(\alpha) \text{ exists} & \Rightarrow & f \text{ has Taylor series expansion near } \alpha \\ \forall \alpha \in \Omega & & \forall \alpha \in \Omega, n \in \mathbb{Z}_{\geq 1} & & \forall \alpha \in \Omega \end{array}$$

All these implications are false in the real case.

$C^1(\mathbb{R}) \subsetneq C^0(\mathbb{R})$ - by taking primitives - $C^{k+1}(\mathbb{R}) \subsetneq C^k(\mathbb{R})$.

e.g. $f(x) = |x|$.

$C^\infty(\mathbb{R}) \supsetneq An(\mathbb{R})$ (analytic functions)

e.g. $f(x) = e^{-\frac{1}{x^2}}$; $f^{(n)}(0) = 0 \ \forall n \geq 0$; $f \in C^\infty(\mathbb{R})$

but $f \not\equiv 0$ in a neighbourhood of 0.

§3. Liouville's theorem* and the fundamental theorem of algebra.

Let $f : \mathbb{C} \to \mathbb{C}$ be a bounded (i.e., $\exists M \in \mathbb{R}_{>0}$ s.t. $|f(z)| < M, \forall z \in \mathbb{C}$) holomorphic function. Then f is constant (i.e. $f(z) = C, \forall z \in \mathbb{C}$)

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* Joseph Liouville (1809-1882). This theorem is actually due to Cauchy (1844). It was called Liouville's thm by Borchardt - who learnt it from Liouville in 1847.

Proof. - Let $z_0 \in \mathbb{C}$. Then, by Cauchy's integral formula

$$f(z_0) - f(0) = \frac{1}{2\pi i} \int_{C(0;R)} \left(\frac{f(z)}{z - z_0} - \frac{f(z)}{z} \right) dz ,$$

for every $R > |z_0|$. So,

$$|f(z_0) - f(0)| = \frac{1}{2\pi} \left| \int_{C(0;R)} \frac{z_0 f(z)}{z(z-z_0)} dz \right|$$

$$< \frac{1}{2\pi} \frac{M \cdot |z_0|}{R(R-|z_0|)} \cdot 2\pi R \to 0 \text{ as } R \to \infty .$$

So, $f(z_0) - f(0) = 0$ showing that $f(z_0) = f(0)$ $\forall z_0 \in \mathbb{C}$. □

Fundamental theorem of algebra. - Let $p(z) \in \mathbb{C}[z]$ be a polynomial of degree $n \geq 1$. Then $p(z) = 0$ has a solution in $\mathbb{C}$.

Proof. - Assume the contrary. Write $p(z) = z^n + a_{n-1} z^{n-1} + \dots + a_0$.
then, by triangle inequality

$$|p(z)| \geq |z|^n - \left(\sum_{j=0}^{n-1} |z|^j \, |a_{n-j}| \right)$$

$$= |z|^n \left(1 - \sum_{j=0}^{n-1} \frac{|a_{n-j}|}{|z|^{n-j}} \right)$$

For $R \gg 0$ s.t. $1 - \sum_{j=0}^{n-1} \frac{|a_{n-j}|}{R^{n-j}} \leq \frac{1}{2}$, we get

$|p(z)| \geq \frac{R^N}{2}$. Hence $\frac{1}{|p(z)|} \leq \frac{2}{R^N}$ for $|z| > R$.

As $\frac{1}{|p(z)|} : \overline{D}(0,R) \to \mathbb{R}_{>0}$ is continuous, it takes a maximum value - so $\frac{1}{p(z)} : \mathbb{C} \to \mathbb{C}$ is a bounded, holomorphic function, hence a constant, contradicting $\deg(p(z)) \geq 1$.

$\square$

<u>Some Historical remarks.</u>– (1) FTA (abbreviation for fundamental theorem of algebra) was conjectured by Albert Girard in 1629. Leibniz did not believe it as he failed to realize

$$\sqrt{i} \text{ exists in } \mathbb{C} \text{ and } \sqrt{i} = \frac{1 \pm i}{\sqrt{2}}.$$

(2) The first, almost successful, attempt on FTA was made by d'Alembert in 1746. His argument used max.- modulus-principle - which follows from Cauchy's theorem. We will revisit it once we have proved Max/min modulus principles.

(3) In 1749, Euler gave a proof of FTA - expanded upon by Lagrange in 1772 - based on induction on n- where $\deg(p(z)) = 2^n \cdot m$ (m: odd). The proof used the fact that

splitting extensions exist - now a theorem due to Kronecker and Kummer, proved, in the later half of nineteenth century.

(4) In 1799, Gauss gave another proof of FTA - based on "intermediate value theorem" - proved much later by Bolzano - Weierstrass.

<u>Gauss' argument</u>. for some $R \gg 0$, $|p(z)| \sim |z^n|$ for $|z| > R$.

Solutions to $z^n = 0$, or $Re(z^n) = 0$ solid lines
$Im(z^n) = 0$ dotted lines

have n components - which alternately touch $|z| = R$ circle

Within $D(0;R)$ the solid lines have to be joined to solid and dotted to dotted

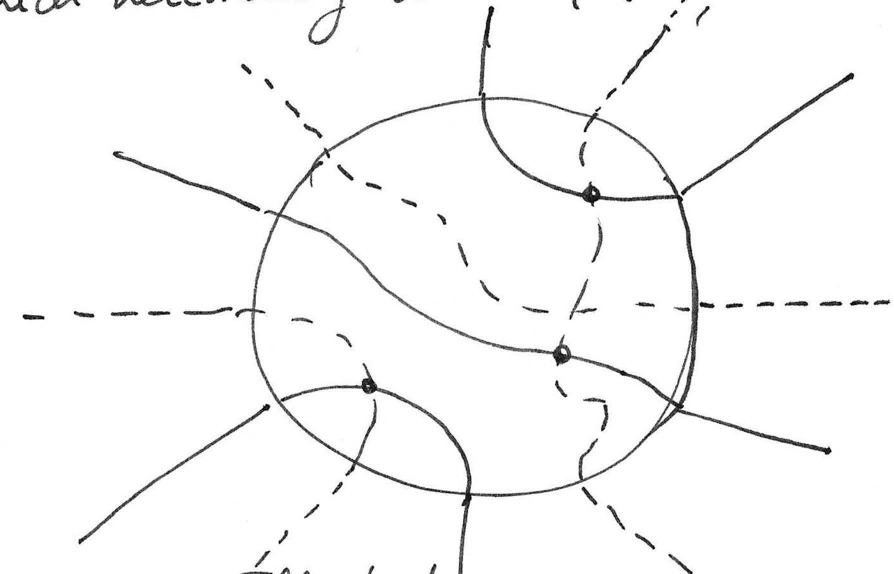

Illustration $n=3$

Hence some solid line must meet a dotted line (Bolzano - Weierstrass)

- at that point $p(z) = 0$.