Lecture 5

Recall that we have proved the following results.

$\Omega \subset \mathbb{C}$ open, connected $\qquad f : \Omega \to \mathbb{C}$ continuous.

f is $\mathbb{C}$-differentiable $\Rightarrow$ Cauchy's theorem and integral formula

$$f(\alpha) = \frac{1}{2\pi i} \int_{\circlearrowleft} \frac{f(s)}{s-\alpha}\, ds$$

The integral formula, and the triangle inequality

$\Rightarrow$ Liouville's theorem and the fundamental theorem of algebra

§1.

$$\boxed{f'(\alpha) = \frac{1}{2\pi i} \int_{C(\alpha;R)} \frac{f(s)}{(s-\alpha)^2}\, ds}$$

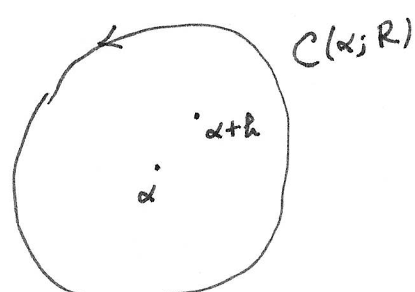
- $f : \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable
- $\alpha \in \Omega$.
- $R \in \mathbb{R}_{>0}$ is such that $\overline{D(\alpha;R)} \subset \Omega$

(closed disc of radius R centered at α)

Proof.- Let $h \in \mathbb{C}$ be s.t. $|h| < R$.

Then, by Cauchy's integral formula

$$f(\alpha+h) - f(\alpha) = \frac{1}{2\pi i} \int_{C(\alpha;R)} f(s) \left(\frac{1}{s-\alpha-h} - \frac{1}{s-\alpha} \right) ds$$



$$\Rightarrow \quad \frac{f(\alpha+h) - f(\alpha)}{h} = \frac{1}{2\pi i} \int_{C(\alpha;R)} \frac{f(s)}{(s-\alpha-h)(s-\alpha)}\, ds$$

Hence $\dfrac{f(\alpha+h)-f(\alpha)}{h} - \dfrac{1}{2\pi i}\displaystyle\int_{C(\alpha,R)} \frac{f(s)}{(s-\alpha)^2}\,ds$

$$= \frac{1}{2\pi i}\int_{C(\alpha;R)} f(s)\left(\frac{1}{(s-\alpha-h)(s-\alpha)} - \frac{1}{(s-\alpha)^2}\right) ds$$

$$= \frac{1}{2\pi i}\int_{C(\alpha;R)} \frac{h\cdot f(s)}{(s-\alpha)^2(s-\alpha-h)}\,ds.$$ By triangle inequality,

$M = \max\{|f(s)| : s\in C(\alpha;R)\}$

$$\left|\frac{f(\alpha+h)-f(\alpha)}{h} - \frac{1}{2\pi i}\int_{C(\alpha;R)}\frac{f(s)}{(s-\alpha)^2}\,ds\right| \leq \frac{|h|\cdot M}{2\pi R^2(R-|h|)}\cdot 2\pi R$$

$\to 0$ as $h\to 0$. □

Remark. - The reasoning given above applies equally well to any $n \geq 1$ and gives

$$\boxed{\frac{f^{(n)}(\alpha)}{n!} = \frac{1}{2\pi i}\int_{\circlearrowleft \alpha} \frac{f(s)}{(s-\alpha)^{n+1}}\,ds}$$

From now on, $\mathbb{C}$-diff. fns. will be called <u>holomorphic</u>.

§2. Uniform convergence and Weierstrass' theorem*

Definition. Let $\Omega \subset \mathbb{C}$ be an open, connected set and let $\{f_n : \Omega \to \mathbb{C}\}_{n=0}^{\infty}$ be a sequence of functions.

We say $\{f_n\}$ converges pointwise if $\forall z \in \Omega$, $\{f_n(z)\}_{n=0}^{\infty}$ is a convergent sequence of complex numbers. That is,

$\forall \varepsilon > 0$, $\exists N > 0$ s.t. $|f_n(z) - f_m(z)| < \varepsilon$ for all $n, m \geq N$.

Let $f(z) := \lim_{n \to \infty} f_n(z)$. We say $f_n \to f$ uniformly with respect to $A \subset \Omega$ if for every $\varepsilon > 0$, there exists $N > 0$ such that

$|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N$ and $z \in A$.

We are interested in uniform convergence w.r.t. compact subsets in Ω.

Thus

$f_n \to f$ uniformly read: f_n converges to f uniformly	means	that the convergence is uniform with respect to K; for any compact $K \subset \Omega$.

Using Heine-Borel theorem, it is equivalent to uniform convergence w.r.t. closed discs in Ω

Theorem. - Let $\{f_n\}$ be a sequence of functions converging uniformly to f.

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* Karl Weierstrass (1815-1897)

(1) If each f_n is continuous, then so is f. In this case,

for any path $\gamma : [0,1] \to \Omega$, we have

$$\lim_{n\to\infty} \int_\gamma f_n(s)\,ds = \int_\gamma f(s)\,ds$$

(2) If each f_n is holomorphic, then so is f and

$\{f_n'\}$ converges uniformly to f'.

Proof (1) Let $z_0 \in \Omega$ and let us prove f is continuous at z_0.

That is, we are given an $\varepsilon > 0$, and we have to find $\delta > 0$

so that $\quad 0 < |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon.$

- Pick $R > 0$ s.t. $\overline{D(z_0; R)} \subset \Omega$. As $f_n \to f$ is uniform

  we can find $N > 0$ s.t. $|f_N(z) - f(z)| < \frac{\varepsilon}{3}$ for every $z \in \overline{D(z_0;R)}$.

- Since f_N is continuous at z_0, we can find $0 < \delta < R$ so that

$$|z - z_0| < \delta \Rightarrow |f_N(z) - f_N(z_0)| < \frac{\varepsilon}{3}$$

So,
$$\begin{aligned}|f(z) - f(z_0)| &\leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| \\ &\quad + |f_N(z_0) - f(z_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \quad \forall\, z \text{ such that } |z - z_0| < \delta.\end{aligned}$$

Hence f is continuous at z_0.

Now, let $\gamma : [0,1] \to \Omega$ be a path. Let $\varepsilon > 0$ be given.

As $\text{image}(\gamma) \subset \Omega$ is compact, by defn. of uniform convergence, we can find $N > 0$ s.t. $|f_n(s) - f(s)| < \frac{\varepsilon}{L}$, $\forall\ n \geq N$ & $s \in \text{image}(\gamma)$.

($L = \text{length}(\gamma)$)

Then,
$$\left| \int_\gamma (f_n(s) - f(s))\, ds \right| < \frac{\varepsilon}{L} \cdot L = \varepsilon.$$

<u>Proof of (2)</u> : We begin by showing that f is holomorphic.

Let $\alpha \in \Omega$ and $r > 0$ be such that $\overline{D(\alpha; r)} \subset \Omega$. Then, for any $h \in \mathbb{C}$, with $|h| < r$ we have

$$\frac{1}{h}\left(f(\alpha+h) - f(\alpha)\right) = \lim_{n\to\infty} \frac{f_n(\alpha+h) - f_n(\alpha)}{h}$$

$$= \frac{1}{2\pi i h} \lim_{n\to\infty} \int_{C(\alpha;r)} f_n(s)\left(\frac{1}{s-\alpha-h} - \frac{1}{s-\alpha}\right) ds$$

$$= \frac{1}{2\pi i \cancel{h}} \lim_{n\to\infty} \int_{C(\alpha;r)} \frac{\cancel{h} f_n(s)}{(s-\alpha-h)(s-\alpha)}\, ds.$$

<u>Check</u>: $$\lim_{n\to\infty} \int_{C(\alpha;r)} \frac{f_n(s)}{(s-\alpha-h)(s-\alpha)}\, ds = \int_{C(\alpha;r)} \frac{f(s)}{(s-\alpha-h)(s-\alpha)}\, ds$$, and

$$\lim_{h \to 0} \int_{C(\alpha;r)} \frac{f(s)}{(s-\alpha-h)(s-\alpha)} \, ds = \int_{C(\alpha;r)} \frac{f(s)}{(s-\alpha)^2} \, ds.$$

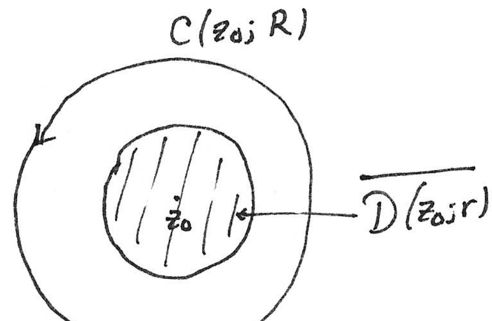
Hence $f'(\alpha)$ exists $\forall \alpha \in \Omega$ and is equal to $\displaystyle\int_{C(\alpha;r)} \frac{f(s)}{(s-\alpha)^2} \, ds.$

Next, we prove that $\{f_n'\}_{n=0}^{\infty}$ converges uniformly to f'.

<u>To prove:</u> Given a compact set $K \subset \Omega$ and $\varepsilon > 0$, we can find $N > 0$ s.t. $|f_n'(z) - f'(z)| < \varepsilon$; $\forall z \in K$ and $n \geq N$.

WLOG, we may assume $K = \overline{D(z_0; r)}$ (a closed disc in Ω).

Pick $R > r$ s.t. $\overline{D(z_0; R)} \subset \Omega$.



Then, for every $z \in K$, we have

$$f_n'(z) - f'(z) = \frac{1}{2\pi i} \int_{C(z_0;R)} \frac{f_n(s) - f(s)}{(s-z)^2} \, ds$$

by triangle inequality, modulus of this term $\displaystyle \leq \frac{\max_{s \in C(z_0;R)} |f_n(s) - f(s)|}{(R-r)^2} \cdot R$

So, (using the defn. of uniform convergence), pick $N > 0$ so that

$$|f_n(s) - f(s)| < \varepsilon \cdot \frac{(R-r)^2}{R}$$

and we are done. $\square$