Recall: Let $\Omega \subset \mathbb{C}$ be an open, connected set, $f : \Omega \to \mathbb{C}$ a holomorphic function. Then

(Cauchy's integral formula)
$$\boxed{\frac{f^{(n)}(\alpha)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-\alpha)^{n+1}}\,ds}$$
$\gamma : [0,1] \to \Omega$ a contour s.t. $\alpha \in$ Interior$(\gamma) \subset \Omega$.

A sequence of functions $\{f_n : \Omega \to \mathbb{C}\}_{n=1}^{\infty}$ is said to <u>converge uniformly to $f : \Omega \to \mathbb{C}$</u> ($\lim_{n\to\infty} f_n = f$ uniformly) if

for any compact set $K \subset \Omega$ and $\varepsilon > 0$, we can find $N > 0$ s.t.
$$|f_n(z) - f(z)| < \varepsilon \quad \text{for each } n \geq N \text{ and } z \in K.$$

[In this case, f_n continuous $\forall n \Rightarrow f$ is continuous and
$$\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz.$$
If each f_n is holomorphic, then so is f and $f_n' \to f'$ uniformly.]
(Weierstrass' thm)

(6.1) Power series and Abel's theorem

$\mathbb{C}[\![z]\!]$ = ring of (formal) power series
$$= \left\{ \sum_{j=0}^{\infty} c_j z^j \;:\; c_j \in \mathbb{C} \text{ for each } j \in \mathbb{Z}_{\geq 0} \right\}.$$

Algebraic operations on $\mathbb{C}[[z]]$:

Addition: $$\sum_{j=0}^{\infty} c_j z^j + \sum_{j=0}^{\infty} d_j z^j = \sum_{j=0}^{\infty} (c_j + d_j) z^j$$

Multiplication: $$\left(\sum_{j=0}^{\infty} c_j z^j\right)\left(\sum_{k=0}^{\infty} d_k z^k\right) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} c_j d_{n-j}\right) z^n$$

Composition: If $p(z) = \sum_{n=0}^{\infty} p_n z^n$ and $q(z) = \sum_{m=1}^{\infty} q_m z^m$, then

$(q(0) = 0)$

$$p(q(z)) = \sum_{n=0}^{\infty} p_n (q(z))^n$$ is again a power series.

Ex. Write the coeff. of z^N in $p(q(z))$ as a polynomial in $p_0, p_1, \dots, p_N$ and $q_1, \dots, q_N$.

Theorem (Abel*) Given $p(z) = \sum_{n=0}^{\infty} p_n z^n \in \mathbb{C}[[z]]$, there exists a unique $R \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that

(1) $\left\{ \sum_{n=0}^{N} p_n z^n \right\}_{N=0}^{\infty}$ converges, uniformly and absolutely on $D(0; R)$.

(2) $\forall z_0 \in \mathbb{C}$ s.t. $|z_0| > R$, $\sum_{n=0}^{\infty} p_n z_0^n$ diverges.

This R is called the radius of convergence of $p(z)$.

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* Niels Henrick Abel (1802-1829)

Proof. Consider the set

$$I = \{ t \in \mathbb{R}_{\geq 0} : \text{there exists } M > 0 \text{ so that } |p_n| \cdot t^n < M \text{ for all } n \geq 0 \}.$$

Note: $0 \in I$, so $I \neq \phi$. I is an interval since $t \in I$ and $s < t$ implies $s \in I$.

Take $R = \sup(I)$.

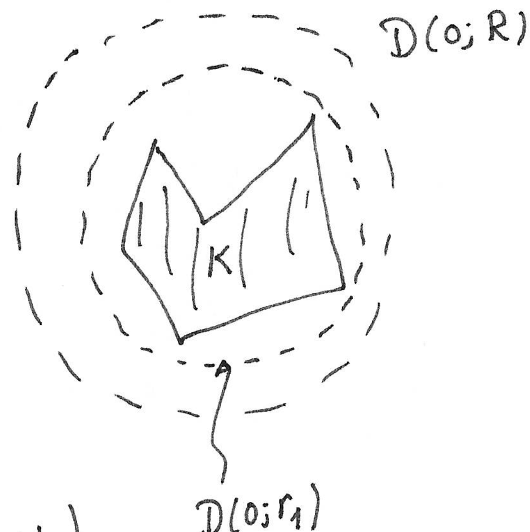
Proof of (2). If $|z_0| > R$, then by defn. of R, $\{ |p_n| \cdot |z_0|^n \}_{n=0}^{\infty}$ is unbounded.

(Recall: $\sum_{n=0}^{\infty} a_n$ converges $\Rightarrow$ $\lim_{n\to\infty} |a_n| = 0$.)

Hence, $\sum_{n=0}^{\infty} p_n z_0^n$ is divergent, as $\lim_{n\to\infty} |p_n z_0^n| \neq 0$.

Proof of (1). If $R = 0$, then $D(0;0) = \phi$, so there is nothing to prove. Assume $R > 0$ and let $K \subset D(0;R)$ be a compact set. Choose $0 < r_1 < R$ such that $K \subset D(0;r_1)$ and let $r_2 \in (r_1, R)$.

As $r_2 < R$, by definition, $\exists\, M > 0$ s.t. $|p_n|\, r_2^n < M$ for every $n \geq 0$.



Thus, $\forall\, z \in K$, we have

$$\sum_{n=0}^{N} |p_n|\, |z|^n < \sum_{n=0}^{N} M \left(\frac{r_1}{r_2}\right)^n$$

$$= M \cdot \frac{1 - t^{N+1}}{1 - t} \qquad \left(t = \frac{r_1}{r_2} < 1\right)$$

Hence, $\sum_{n=0}^{\infty} p_n z^n$ converges uniformly and absolutely (rel. to compact subsets in $D(0;R)$). $\square$

§2. Remarks and examples. - Combining Abel's and Weierstrass' theorems, if $p(z) = \sum_{n=0}^{\infty} p_n z^n$ has non-zero radius of convergence, then it defines a holomorphic function $D(0;R) \to \mathbb{C}$.

Moreover $p'(z) = \sum_{n=1}^{\infty} n p_n z^{n-1}$ uniformly on $D(0;R)$.

By an easy induction argument, one gets

$$\boxed{p_n = \frac{p^{(n)}(0)}{n!}}$$

In practice, one uses either d'Alembert's ratio test, or Hadamard's root test formula to get the radius of convergence.

(Ratio) If $\ell = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$ exists, then $R = \frac{1}{\ell}$ $(= \infty \text{ if } \ell = 0)$.

(assume $c_n \neq 0$)

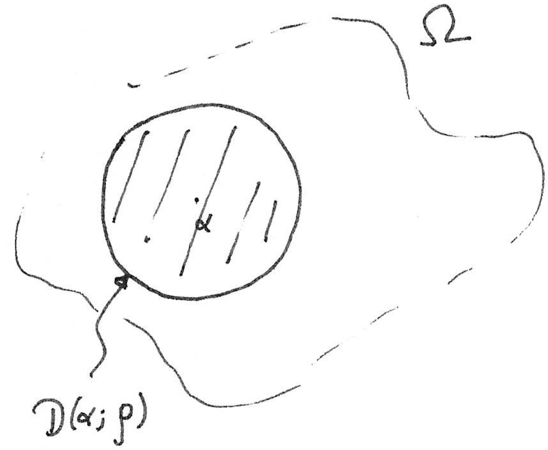
(Root test formula) $$\boxed{\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}}$$

e.g. $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ on $D(0,\infty) = \mathbb{C}$.

$$\sum_{n=0}^{\infty} n^k z^n \quad (k \in \mathbb{Z}_{\geq 0}) \quad \text{has radius of convergence} = 1.$$

$$\sum_{n=0}^{\infty} n!\, z^n \quad \text{has radius of convergence } 0.$$

§3. Taylor* series. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function, $\alpha \in \Omega$ and $\rho > 0$ be such that $\overline{D(\alpha, \rho)} \subset \Omega$.



Theorem. - There is a unique power series $F(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ of radius of convergence $\geq \rho$ s.t. $f(z) = F(z) \ \forall z \in D(\alpha; \rho)$.

The coefficients $c_0, c_1, \ldots$ are given by

$$c_n = \frac{1}{2\pi i} \int_{\mathcal{C}(\alpha, \rho)} \frac{f(\xi)}{(s-\alpha)^{n+1}}\, ds = \frac{f^{(n)}(\alpha)}{n!}.$$

(by Cauchy's formula)

Proof.- Let $F(z)$ be the power series (centered at α) given as above

$$F(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n \ ; \quad c_n := \frac{1}{2\pi i} \int_{\mathcal{C}(\alpha; \rho)} \frac{f(s)}{(s-\alpha)^{n+1}}\, ds.$$

---

* Brook Taylor (1685-1731)

Claim. – Radius of convergence of F is $\geq \rho$.

By the proof of Abel's theorem, it suffices to find $M>0$ so that $|c_n| \cdot \rho^n < M$.

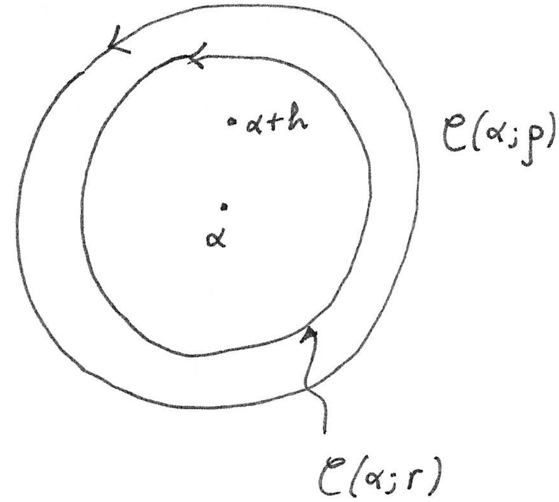
Take $M > \operatorname{Max}\{|f(s)| : s \in C(\alpha;\rho)\}$. Then, by triangle inequality,

$$|c_n| = \left| \frac{1}{2\pi i} \int_{C(\alpha;\rho)} \frac{f(s)}{(s-\alpha)^{n+1}}\, ds \right| < \frac{1}{2\pi} \frac{M}{\rho^{n+1}} 2\pi\rho = \frac{M}{\rho^n}.$$

Next, we show that $F(z) = f(z) \quad \forall\, z \in D(\alpha;\rho)$.

Write $z = \alpha + h$ so that $|h| < \rho$.

Choose $r \in (|h|, \rho)$.



By Cauchy's formula

$$f(\alpha+h) = \frac{1}{2\pi i} \int_{C(\alpha;r)} \frac{f(s)}{s-\alpha-h}\, ds$$

for $s \in C(\alpha;r)$, $|s-\alpha| = r > |h|$, so

$$\frac{1}{s-\alpha-h} = \frac{1}{s-\alpha} \frac{1}{1-\frac{h}{s-\alpha}}$$

$$= \sum_{n=0}^{\infty} \frac{h^n}{(s-\alpha)^{n+1}} \quad \text{uniformly}$$

So, $$f(\alpha+h) = \frac{1}{2\pi i} \int_{C(\alpha;r)} \left(f(s) \cdot \sum_{n=0}^{\infty} \frac{h^n}{(s-\alpha)^{n+1}} \right) ds$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C(\alpha;r)} \frac{f(s)}{(s-\alpha)^{n+1}} ds \right) \cdot h^n \qquad (\text{recall: } h = z-\alpha)$$

$$= F(\alpha+h).$$

□

§4. Isolated singularity and Laurent* series.

Assume: $f : \Omega \to \mathbb{C}$ is holomorphic., $\alpha \in \mathbb{C}$, $\alpha \notin \Omega$

and there exists $\rho > 0$ s.t.

$$D^{\times}(\alpha;\rho) := D(\alpha;\rho) \setminus \{\alpha\} \subset \Omega.$$

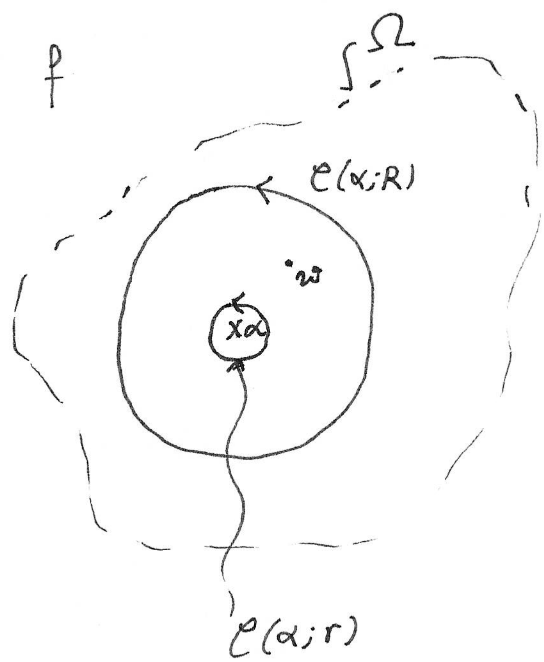
We say α is an isolated singularity of f

Notation: $Ann(\alpha; r<R)$

$= \{z : r < |z-\alpha| < R\}$

annular neighbourhood of α.

Choose $0 < r < R$ s.t. $\overline{Ann(\alpha; r<R)} \subset \Omega$



Theorem: There are two unique power series

$F^{+}(z) \in \mathbb{C}[\![z-\alpha]\!]$ and

$F^{-}(z) \in \mathbb{C}[\![(z-\alpha)^{-1}]\!]$

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* Pierre Alphonse Laurent (1813-1854)

$$F^+(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n \quad ; \quad F^-(z) = \sum_{m=1}^{\infty} d_m (z-\alpha)^{-m}$$

such that $f(z) = F^+(z) + F^-(z)$ uniformly (on cpct subsets of) $D^\times(\alpha; R)$.

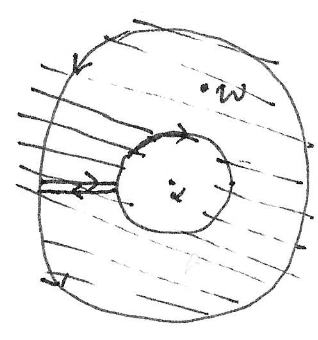
$F^+(z)$ has radius of convergence $\geq R$; and

$F^-(z) : \mathbb{C} \setminus \{\alpha\} \to \mathbb{C}$ holomorphic

$$c_n = \frac{1}{2\pi i} \int_{\mathcal{C}(\alpha;R)} \frac{f(s)}{(s-\alpha)^{n+1}}\, ds \qquad d_m = \frac{1}{2\pi i} \int_{\mathcal{C}(\alpha;r)} f(s) \cdot (s-\alpha)^{m-1}\, ds$$

Proof. Let $w \in \mathrm{Ann}(\alpha; r < R)$. Then by Cauchy's formula

$$f(w) = \frac{1}{2\pi i} \int_{\mathcal{C}(\alpha;R) - \mathcal{C}(\alpha;r)} \frac{f(s)}{s-w}\, ds$$



Shaded part $\subset \Omega$.

$\mathcal{C}(\alpha;R) - \mathcal{C}(\alpha;r)$

Set $F^+(w) = \frac{1}{2\pi i} \int_{\mathcal{C}(\alpha;R)} \frac{f(s)}{s-w}\, ds$ for $w \in D(\alpha;R)$

$$= \frac{1}{2\pi i} \int_{\mathcal{C}(\alpha;R)} \frac{f(s)}{(s-\alpha)-(w-\alpha)}\, ds = \sum_{n=0}^{\infty} (w-\alpha)^n \cdot \left(\frac{1}{2\pi i} \int_{\mathcal{C}(\alpha;R)} \frac{f(s)}{(s-\alpha)^{n+1}}\, ds \right)$$

(same calculation & unif. convergence of geometric series) $\leftarrow$ uses $|w-\alpha| < R = |s-\alpha|$.

$$F^-(w) = \frac{-1}{2\pi i} \int_{\mathcal{C}(\alpha;r)} \frac{f(s)}{s-w}\, ds = \frac{-1}{2\pi i} \int_{\mathcal{C}(\alpha;r)} \frac{f(s)}{(s-\alpha)-(w-\alpha)}\, ds$$

$$= \frac{1}{2\pi i} \int_{\mathcal{C}(\alpha;r)} \frac{f(s)}{(w-\alpha)-(s-\alpha)}\, ds$$

for $w \in \mathbb{C} \setminus \overline{D(\alpha;r)}$

$|w-\alpha| > r = |s-\alpha|$

$$= \sum_{n=0}^{\infty} (w-\alpha)^{-n-1} \cdot \left(\oint_{\mathcal{C}(\alpha;r)} f(s) \cdot (s-\alpha)^n \, ds \right)$$

[Notation: $\displaystyle\oint_C = \frac{1}{2\pi i} \int_C$]

Note: as $r>0$ was arbitrary - the same calculation (as in Taylor series) shows that

$F^- : \mathbb{C} \setminus \{\alpha\} \to \mathbb{C}$ (i.e., has infinite radius of convergence near ∞.)

$F^-(\infty) = 0.$

$\square$

§5. Types of singularities. Again $\alpha \notin \Omega$ is a singularity (isolated) of f (i.e. $D^\times(\alpha;\rho) \subset \Omega$ for some $\rho > 0$).

- Removable. if all d_n's are zero. (from Laurent's thm. above) Thus, $f(z) = F^+(z)$ is defined on $D(\alpha;R)$.

- Pole of order n. ($n \geq 1$) if $d_{n+1} = d_{n+2} = \cdots = 0$ and $d_n \neq 0$. Thus, $f(z) = F^+(z) + ($poly of deg n in $(z-\alpha)^{-1})$

- Essential singularity: $\{n : d_n \neq 0\}$ is infinite.

  (non-isolated singularities are also considered essential).