Lecture 7

§1. Meromorphic function. ('mero' = part; 'holo' = whole).

$f : \Omega \dashrightarrow \mathbb{C}$ is a meromorphic function, if $\exists$ a ~~isolated~~ set of isolated points $A \subset \Omega$ (i.e., $\forall a \in A$, $\exists\, r > 0$ s.t. $D^*(a,r) \cap A = \phi$); so that (i) $f : \Omega \setminus A \to \mathbb{C}$ is holomorphic.

(ii) Every $a \in A$ is either a removable singularity, or a pole of f.

Recall: if $\alpha \in \mathbb{C}$ is an isolated singularity of f, then we can write

$$f(z) = \underbrace{\sum_{m=1}^{\infty} d_m (z-\alpha)^{-m}}_{\text{singular part of } f} + \underbrace{\sum_{n=0}^{\infty} c_n (z-\alpha)^{n}}_{\text{regular part of } f}$$

We say α is removable if $d_1 = d_2 = \ldots = 0$ (i.e. singular part $= 0$).

pole of order N if $d_{N+1} = d_{N+2} = \cdots = 0$; $d_N \neq 0$. $(N \geq 1)$.

essential singularity if $\{m : d_m \neq 0\}$ is infinite.

Exercise. Show that α is removable $\iff$ $\lim_{z \to \alpha} |f(z)|$ exists & is finite.

α is a pole $\iff$ $\lim_{z \to \alpha} |f(z)| = \infty$.

α is essential $\iff$ $\lim_{z \to \alpha} |f(z)|$ does not exist.

§2. Point at infinity.

Behaviour of a mero. fn. $f : \mathbb{C} \dashrightarrow \mathbb{C}$ near $z = \infty$ is, by defn, same as the behaviour of $f\left(\frac{1}{w}\right)$ near $w = 0$.

Prop. – (1) Assume $f: \mathbb{C} \to \mathbb{C}$ is holomorphic. If f has a removable singularity (resp. pole of order n) near ∞, then f is a constant function (resp. polynomial of deg. n).

(2) Assume $f: \mathbb{C} \dashrightarrow \mathbb{C}$ is meromorphic. If f has either removable singularity or a pole of order n at ∞, then

$$f(z) = \frac{P(z)}{Q(z)} \; ; \; P, Q \in \mathbb{C}[z]. \qquad \begin{array}{l} \deg(P) \leq \deg(Q) \text{ for removable} \\ \deg(P) = \deg(Q) + n \text{ for pole of ord } n. \end{array}$$

(rational function)

Proof. Assume we have a mero. fn. $g: \mathbb{C} \dashrightarrow \mathbb{C}$ for which ∞ is an isolated singularity. Let $A \subset \mathbb{C}$ be the set of poles of g. Then $|A| < \infty$. This is because, there exists $R > 0$ so that

$D^{\times}(\infty; R) = \{z \in \mathbb{C} : |z| > R\} \subset$ domain of g

i.e., $A \subset \overline{D(0; R)}$. Being a discrete set in a compact one, A must be finite.

As each $\alpha \in A$ is a pole of g, its singular part near α is a rational function, vanishing at ∞.

$$g(z) - \sum_{\alpha \in A} \text{Singular part of } g \text{ near } \alpha \; : \; \mathbb{C} \to \mathbb{C} \text{ is holomorphic}$$

and has the same nature of singularity at ∞, as g does.

If g has a pole of order n near ∞, then $\lim\limits_{z \to \infty} z^{-n} g(z)$ exists.

Cancelling the rational singular part of $z^{-n} g(z)$,

$$h(z) = z^{-n} g(z) - \sum_{a \in A \cup \{0\}} \text{singular part of } z^{-n} g(z) \text{ near } a$$

we get $h : \mathbb{C} \to \mathbb{C}$ holomorphic, and $\lim_{z \to \infty} |h(z)|$ exists.

$\Rightarrow$ h is bounded, and hence by Liouville's theorem, a constant. □

§3. Identity theorem; or principle of permanence of relations.

Let $f : \Omega \to \mathbb{C}$ be a holomorphic function on an open, connected set $\Omega \subset \mathbb{C}$.

Theorem.- If $\exists \{z_n\}_{n=1}^{\infty} \subset \Omega$ converging to $\alpha \in \Omega$ such that $f(z_n) = 0 \;\forall n \geq 1$. Then $f \equiv 0$ on Ω.

Proof.- Given any point $\beta \in \Omega$, by Taylor series expansion, either there exists $r > 0$ such that $f(z) = \sum_{n=N}^{\infty} c_n (z-\beta)^n$ ($N \geq 0$ is the smallest s.t. $c_N \neq 0$)

or $f \equiv 0$ on $D(\beta; r)$.

In the first case, $f(z) = (z-\beta)^N \underbrace{\left(c_N + (z-\beta)\, c_{N+1} + \dots \right)}_{h(z)}$

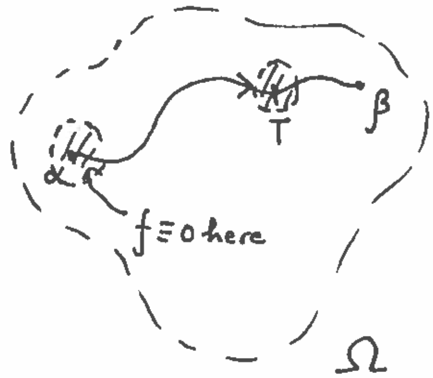
$h(\beta) \neq 0 \Rightarrow \exists\, \rho > 0$ s.t. $h(z) \neq 0 \;\forall z \in D(\beta; \rho)$.

Hence we conclude: for $\beta \in \Omega$, we have two possibilities:

- either $f(z) = 0 \;\forall z \in D(\beta; r)$.
- or $f(z) \neq 0 \;\forall z \in D^{*}(\beta; \rho)$.

(*) [If $\exists \{w_n\} \to \beta$; $f(w_n) = 0 \;\forall n$, then we are not in the second case, implying $f \equiv 0$ in an open disc around β.]

Now, let $\beta \in \Omega$ be arbitrary. Choose a path joining α to β, say



$$\gamma : [0,1] \to \Omega \; ; \; \gamma(0) = \alpha, \; \gamma(1) = \beta.$$

$$\text{Set } T = \operatorname{Sup} \overbrace{\{ 0 \leq t < 1 \; : \; f(\gamma(t)) = 0 \}}^{I}.$$

As $\exists \, \rho > 0$ s.t. $f(z) = 0 \;\; \forall z \in D(\alpha; \rho)$, we know $T > 0$.

<u>Claim</u>: $T = 1$; hence $f(\beta) = f(\gamma(1)) = 0$.

<u>Proof</u>. If $T < 1$, then for each $n \geq 1$, we can find $t_n \in I$ s.t.

$$\gamma(t_n) \in D(\gamma(T); \tfrac{1}{n}) \qquad f(\gamma(t_n)) = 0.$$

$\Rightarrow$ by $(*)$ on the previous page, that $\exists \, r > 0$ s.t. $f(z) = 0 \;\; \forall$ $z \in D(\gamma(T); r)$. So, we can find $\varepsilon > 0$ s.t. $T + \varepsilon \in I$ contradicting the defn. of supremum. $\square$

§4. <u>Analytic continuation (Weierstrass)</u> .- Let $\alpha \in \Omega$ and let f be a hol. fn. defined in an open set containing α.

Thus, for some $r > 0$, we can view f as a power series on $D(\alpha; r)$

$$f(z) = \sum_{n=0}^{\infty} C_n(f;\alpha) \cdot (z - \alpha)^n \quad ; \quad C_n(f;\alpha) = \frac{f^{(n)}(\alpha)}{n!}$$

Let $\gamma : [0,1] \to \mathbb{C}$ be a path starting at α (i.e., $\gamma(0) = \alpha$).

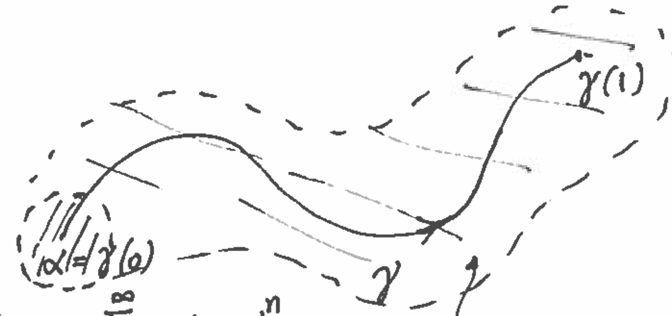
$\boxed{An_\gamma(f)}$ = analytic continuation of f along γ is defined as:
(it may not exist!)

$An_\gamma(f)$ exists if

- $\exists\ \mathcal{U} \subset \mathbb{C}$ open set containing
  connected
  $\gamma([0,1]) \subset \mathcal{U}$
- $F : \mathcal{U} \to \mathbb{C}$ hol. fn.

such that $F\big|_{D(\alpha;\rho)} = f$

for some $\rho > 0$.



$$f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$$

$\forall\, z \in D(\alpha; r)$

$$\mathcal{U} \xrightarrow{F} \mathbb{C}$$

$An_\gamma(f) =$ Taylor series expansion of F near $\gamma(1)$.

Hence, An_γ : "some power series centered at $\alpha = \gamma(0)$" $\longrightarrow$ "Some power series centered at $\beta = \gamma(1)$".