# Lecture 8

Recall: $\left[\begin{array}{l} \text{if } f_1, f_2 : \Omega \to \mathbb{C} \text{ are two holomorphic functions defined} \\ \text{on an } \cancel{\text{comp}} \text{ open, connected set } \Omega \subset \mathbb{C}, \text{ then} \\ f_1(z) = f_2(z) \ \forall z \in \Omega \quad \Longleftrightarrow \quad \exists \{z_n\}_{n=1}^{\infty} \subset \Omega \text{ converging to } \alpha \in \Omega \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{s.t. } f_1(z_n) = f_2(z_n) \ \forall n \geq 1. \end{array}\right.$

(principle of permanence of relations).

§1. Let us revisit our definition of analytic continuation.

<u>Defn.</u> Let $\alpha \in \mathbb{C}$. The <u>ring of germs of holomorphic functions near $\alpha$</u>, denoted by $\mathcal{O}_\alpha$, is defined as:

$$\mathcal{O}_\alpha = \frac{\left\{ (f, \mathcal{U}) \;\middle|\; \begin{array}{l} \mathcal{U} \subset \mathbb{C} \text{ is an open set containing } \alpha \\ f : \mathcal{U} \to \mathbb{C} \text{ is a holomorphic function} \end{array} \right\}}{\text{Equivalence rel}^n : \ (f, \mathcal{U}) \sim (g, \mathcal{V}) \text{ if } f\big|_{D(\alpha;r)} = g\big|_{D(\alpha;r)} \text{ for some } r > 0.}$$

Easy exercise. —
$$\mathcal{O}_\alpha = \mathbb{C}\{z-\alpha\} = \left\{ p(z) \in \mathbb{C}[\![z-\alpha]\!] : \begin{array}{l} \text{radius of} \\ \text{convergence of} \\ p(z) > 0 \end{array} \right\}$$
$$\subset \mathbb{C}[\![z-\alpha]\!].$$

Given $f \in \mathcal{O}_\alpha$ and $\underset{\gamma(0)=\alpha}{\gamma : [0,1] \to \mathbb{C}}$; we say <u>$f$ can be analytically continued along $\gamma$</u>, if $\exists$ an open set $\mathcal{U} \subset \mathbb{C}$, $\gamma([0,1]) \subset \mathcal{U}$ and a hol. fn. $F : \mathcal{U} \to \mathbb{C}$ such that $F\big|_{D(\alpha;r)} = f$ for some $r > 0$.

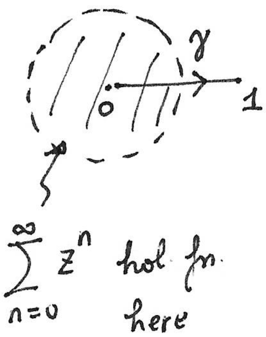
Define $\Omega_f = \{ \beta \in \mathbb{C} \mid \exists\, \gamma : [0,1] \to \mathbb{C}$ s.t. $\gamma(1) = \beta;\ \gamma(0) = \alpha$ & $f$ can be analytically continued along $\gamma \}$

$\subset \mathbb{C}$.

<u>Easy fact</u> : $\Omega_f \subset \mathbb{C}$ is open and connected.

<u>Examples.</u> (1) $\displaystyle\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$.

$\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = t$.




If $\mathcal{A} = \{ T \in [0,1)$ s.t. $\sum z^n$ can be analytically continued along $\gamma|_{[0,T]} \}$

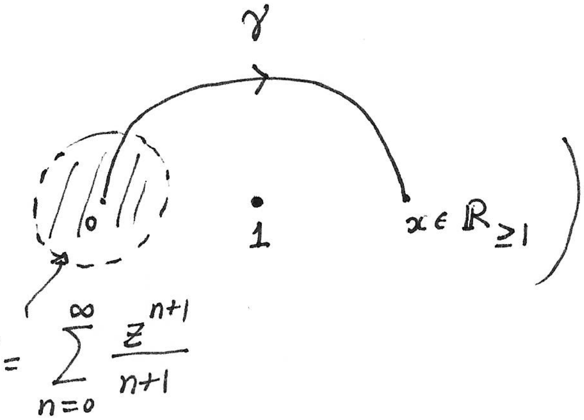
then $\mathcal{A} = [0,1)$, $\sup \mathcal{A} = 1$ and $\gamma(1) = 1$ is a <u>barrier</u>.

$\Omega_f = \mathbb{C} \setminus \{1\}$ $\quad$ 1 is a pole of $f : \Omega_f \dashrightarrow \mathbb{C}$.

(2) $\displaystyle f = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} = \text{“}-\log(1-z)\text{”}$ for $z \notin \mathbb{R}_{\geq 1}$.

so, $\mathbb{C} \setminus \mathbb{R}_{\geq 1} \subset \Omega_f$.

By our defn, $x \in \mathbb{R}_{>1}$ are still in $\Omega_f$ ( $\mathrm{An}_\gamma(f)$ exists – see :

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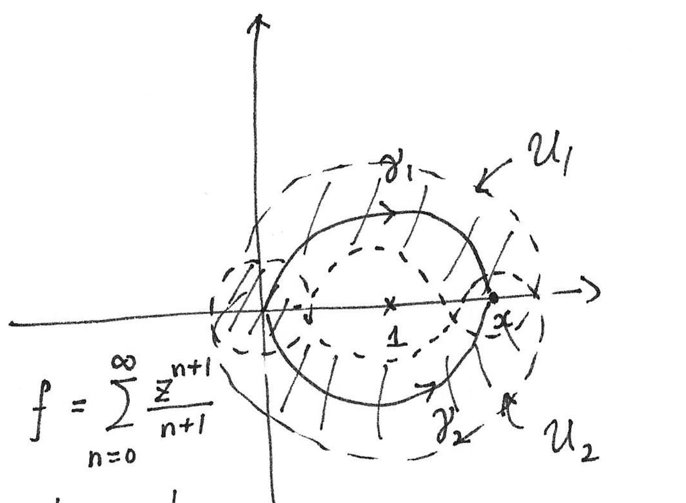
In fact $\Omega_f = \mathbb{C} \setminus \{1\}$ ; but $\Omega_f$ is not the domain of any hol. fn which agrees with $\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}$ near $0$.

In our defn of $\Omega_f$, no uniqueness of path $\gamma$ is assumed. In fact for $\beta \in \Omega_f$ there could be two paths $\gamma_1, \gamma_2 : [0,1] \to \Omega_f$ s.t. joining $\alpha$ & $\beta$

$$An_{\gamma_1}(f) \neq An_{\gamma_2}(f).$$

<u>Calculation</u> -




$$f = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}$$

$$f' = \frac{1}{1-z}$$

Extending functions :

$F_1 : \mathcal{U}_1 \to \mathbb{C}$

$F_2 : \mathcal{U}_2 \to \mathbb{C}$

can be defined by

$$F_j(z) = \int_{\substack{0 \to z \\ \text{in } \mathcal{U}_j}} \frac{1}{1-w}\, dw$$

$\mathcal{U}_1, \mathcal{U}_2$ open sets, simply-connected, containing $\gamma_1$ and $\gamma_2$ resp.

Then $$F_2(x) - F_1(x) = \int_{\gamma_2 \cdot \gamma_1^{-1}} \frac{1}{1-w}\, dw = \int_{\circlearrowleft 1} \frac{-1}{w-1}\, dw = -2\pi i.$$

This also shows that

$$\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} - 2\pi i \ : \text{two different}$$

germs at $0$ can be obtained from each other by analytic continuation.

§2. Starting from $f(z)$, a holomorphic function defined near $\alpha \in \mathbb{C}$, we defined $\Omega_f \subset \mathbb{C}$ an open, connected set; using the notion of analytic continuation.

Proposition. – (Corollary of permanence of relations).

Let $\beta \in \Omega_f$ and $\gamma_1, \gamma_2 : [0,1] \to \Omega_f$ be two $\underbrace{\text{simple}\ \text{paths}}_{\text{non-intersecting}}$ joining $\alpha$ & $\beta$

s.t. $\text{interior}\,(\gamma_2^{-1}\gamma_1) \subset \Omega_f$. (our notion of homotopy)

Then $An_{\gamma_1}(f) = An_{\gamma_2}(f)$.

Riemann Surface of f ~~$\Omega(f$~~ $\widetilde{\Omega}_f \subset \bigsqcup_{a \in \Omega_f} \mathcal{O}_a$ (recall $\mathcal{O}_a = \mathbb{C}\{z-a\}$)

$$\widetilde{\Omega}_f = \left\{ (b, [g]) \;:\; b \in \Omega_f \text{ and } [g] = An_\gamma(f) \text{ for some } \gamma : [0,1] \to \Omega_f \text{ joining } \alpha \text{ to } b \right\}$$

Topology on $\widetilde{\Omega}_f$:

Small neighbourhoods of a point $(b,g) \in \widetilde{\Omega}_f$ are given by:

if $\rho > 0$ is the radius of convergence of $g \in \mathcal{O}_b$

then $\underline{N(b,g;\rho)} = \{ (a, g^{(a)}) : a \in D(b,\rho),\ g^{(a)} \in \mathbb{C}\{z-a\}$ re-expansion of $g$ near $a \}$

Note: Under $p : \widetilde{\Omega}_f \longrightarrow \Omega_f$ ; $N(b,g;\rho) \xrightarrow{\sim} D(b,\rho)$
$(b,g) \longmapsto b$ "local coordinate"

§3. Path and homotopy lifting properties: $f \in \mathcal{O}_\alpha \rightsquigarrow p : \widetilde{\Omega}_f \to \Omega_f$

$\Omega_f \subset \mathbb{C}$ open & connected.

**Prop.**– (1) For any $\gamma : [0,1] \to \Omega_f$ ; $\gamma(0) = \alpha$; and $(\alpha, g) \in \bar{p}^{-1}(\alpha) \subset \tilde{\Omega}_f$ ; there exists a unique path $\tilde{\gamma} : [0,1] \to \tilde{\Omega}_f$ , $\tilde{\gamma}(0) = (\alpha, g)$.

(2) If $\gamma_1, \gamma_2 : [0,1] \to \Omega_f$ are simple, non crossing paths with the same endpoints, s.t Interior $(\gamma_2 \gamma_1^{-1}) \subset \Omega_f$ , then ($\tilde{\gamma}_1$ is homotopic to $\tilde{\gamma}_2$) $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$.

(This is just a restatement of Prop §2 from the previous page).

§4. Natural boundary. – There exist power series which cannot be continued beyond its disc of convergence.

Example. (Weierstrass) $\sum_{n=0}^{\infty} z^{n!}$ or $\sum_{n=0}^{\infty} z^{2^n}$ have radius of convergence 1. But the first diverges for every $a \in S^1$ s.t. $a^n = 1$ for some $n$; the second diverges for every $a \in S^1$ s.t $a^{2^l} = 1$ for some $l$.

$\{ e^{2\pi i x} : x \in \mathbb{Q} \}$ ; $\{ e^{\frac{2\pi i m}{2^l}} : l \geq 0 ; m \text{ odd} \}$ are both dense in $S^1$.

Such series are called lacunary and are defined by growing gaps in successive terms.

Natural boundary (of a power series $f$) is a curve $C$ such that $C \subset \bar{\Omega}_f \setminus \Omega_f$ (assuming such exists).

§5. Our next topic will be the study of functions satisfying some differential / difference equation.

System of equations ⇝ holo/mero morphic function –
- local study (power series expansion)
- global study – analytic continuation (jump behaviour).

Another useful, and related question is to find approximating sequence of functions. Such as:

- Taylor series: $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ $\longleftrightarrow$ $f$ can be approximated (near $\alpha$) by polynomials $\left\{ \sum_{n=0}^{N} c_n (z-\alpha)^n \right\}_{N=0}^{\infty}$
- Infinite product formulae e.g. $\frac{\sin(z)}{z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right)$
- Integral formulae $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt$

→ Approximation by rational functions (Padé)

→ asymptotic expansion near a singular point. (if needed)