§1. <u>Differential Equations.</u> A differential equation of order $N \geq 1$ is an equation of the form

$$f^{(N)}(z) = P(z; f(z), f'(z), \dots, f^{(N-1)}(z)) \quad - \text{(Diff'l Eq}^{n}\text{)}$$

where P is a polynomial in $f, f', \dots, f^{(N-1)}$ with coefficients rational functions of z (the nature of P can be obviously generalized).

e.g. $$f''(z) = f(z)^2 + a(z) \qquad \text{Order 2, degree 2.}$$

$$f^{(3)}(z) = a(z)\, f'(z)^4 + b(z)\, f^{(2)}(z)\, f(z) \quad \dots \quad \text{and so on.}$$

Frobenius method. – is to solve the differential eq^n locally – i.e., as a power series in z (for various algebraic relations, such methods were used earlier by Newton and Puiseaux.). Our goals are:

- To determine the global nature of our solution; ← this is a problem of analytic continuation.

- Find better approximations to this solution.

§2. <u>Linear differential equations.</u> – Some examples

A linear differential eq^n is the one depending linearly on $f, f', \dots, f^{(N)}$.

$$\sum_{j=0}^{N} a_j(z) \frac{d^j}{dz^j}(f) = g \quad - (*)$$

We say $(*)$ is <u>homogeneous</u> if $g \equiv 0$.

Remark. (1) Linear diff'l eq^{n}s are special because of the "superposition principle" : $\left(\text{Let } L = \sum_{j=0}^{N} a_j \partial_z^j \right)$.

$$\begin{aligned} L f_1 &= g \\ L f_2 &= g \end{aligned} \quad \Rightarrow \quad L(f_1 - f_2) = 0 \text{ hgs. eq}^n \quad ;$$

f_1, f_2 two solns. of $Lf = 0$ then $\forall\, \alpha_1 f_1 + \alpha_2 f_2$ is another solution.

(2) Frobenius also observed that

Order N, linear hgs diff'l eq^n for $f : \Omega \to \mathbb{C}$ $\longleftrightarrow$ First order, linear hgs. eq^n for $N \times N$ matrix, whose entries are hol. fns. $\Omega \to \mathbb{C}$

$$f^{(N)}(z) = \sum_{j=0}^{N-1} a_j(z) f^{(j)}(z) \quad \rightsquigarrow \quad \Psi(z) = \begin{bmatrix} f \\ f' \\ \vdots \\ f^{(N-1)} \end{bmatrix} \in \Omega \to \mathbb{C}^N$$

assume $a_0 \not\equiv 0$.

$$\Psi'(z) = A(z) \Psi(z)$$

$$A(z) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \\ a_0 & a_1 & \cdots & & a_{N-1} \end{bmatrix}$$

Companion matrix of $t^N - \sum_{j=0}^{N-1} a_j t^j$

$$\Psi'(z) = \begin{bmatrix} f' \\ f'' \\ \vdots \\ \sum_{j=0}^{N-1} a_j f^{(j)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a_0 & a_1 & \cdots & & a_{N-1} \end{bmatrix} \begin{bmatrix} f \\ f' \\ \vdots \\ f^{(N-1)} \end{bmatrix}$$

§3. Examples. – (1) The simplest example is $f'(z) = g(z)$.

(1^{st} order, linear, inhomogeneous)

Local solution. If $g(z) = \sum_{n=0}^{\infty} g_n (z-\alpha)^n$ near $\alpha \in \Omega$ ($g : \Omega \to \mathbb{C}$)

then $F_\alpha(z) := \sum_{n=0}^{\infty} g_n \frac{(z-\alpha)^{n+1}}{n+1} \in \mathbb{C}\{z-\alpha\}$ is a local solution near α. These are normalized to vanish at $z=\alpha$.

Global solution. For any simply-connected subset containing α, say $U \subset \Omega$, we can define (Choose $\alpha \in \Omega$ "base point")

$$F_U(z) = \int_{\substack{\gamma : [0,1] \to U \\ \gamma(0)=\alpha \\ \gamma(1)=z}} g(w)\, dw .$$

Note : $F_U = F_\alpha$ on $D(\alpha; r)$ for some $r>0$.

For any $\beta \in U$,

$F_U(\alpha) = 0$ ← normalization.

$F_U = F_\beta + F_U(\beta)$ on $D(\beta; r)$ for some $r>0$.

The solution will in general be multivalued.

(2) Second simplest example

$$f'(z) = a(z) f(z).$$

$a : \Omega \to \mathbb{C}$ ($\Omega \subset \mathbb{C}$ open, connected).

Local soln. If $a(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n$ near $\alpha \in \Omega$

then $f(z) = 1 + \sum_{n=1}^{\infty} f_n (z-\alpha)^n$ solves $f' = af$ if and only if :

$$(n+1) f_{n+1} = \sum_{j=0}^{n} a_j f_{n-j} \qquad (f_0 = 1 \text{ normalization}).$$

$(\forall n \geq 0)$.

e.g. $f_0 = 1$; $f_1 = a_0$; $f_2 = \frac{1}{2}(a_0 f_1 + a_1 f_0) = \frac{1}{2}(a_0^2 + a_1)$

and so on.

Ex. If R = radius of convergence of $\sum_{j=0}^{\infty} a_j z^j$, then $f(z)$ defined as above also has radius of convergence $= R > 0$.

So, for any $\alpha \in \Omega$, we have a local solution; let us denote it by $F_\alpha : D(\alpha; r) \to \mathbb{C}$ $\quad (F_\alpha(\alpha) = 1)$.

Global solns:

$$\boxed{f(z) = \exp \int_{\alpha \to z} a(w)\, dw}$$

If $f(z) = \exp(\varphi(z))$ then $f' = f \cdot \varphi'(z)$. So, we first solve

$\varphi' = a$ (remember – it may not be a global function)

$\varphi(\alpha) = 0$ (normalization)

and set $f = \exp(\varphi)$.

§4. First order, linear diff'l eq^n, hgs, matrix-valued.

Let $N \geq 1$. $\boxed{\psi'(z) = A(z)\,\psi(z)}$ $A : \Omega \to M_{N\times N}(\mathbb{C})$

i.e. $A = (a_{ij}(z))$

$a_{ij} : \Omega \to \mathbb{C}$ hol.

or mero.

Local Study $\alpha \in \Omega$ isolated sing. of A

$A : \Omega \dashrightarrow M_{N\times N}(\mathbb{C})$ mero.

We say α is a regular point if A is hol. at α.

——— regular singular point (or Fuchsian) if A has a pole of order 1 at α. (Simple pole)

——— irregular point (or Poincaré rank r) if A has a pole of ord. $r \geq 2$ at α

Example of irregular case. –

$$\varphi'(z) = \frac{\varphi(z)}{z^2} - \frac{1}{z}$$ can be solved formally

near 0 ; $\varphi(z) = z + \sum_{n=2}^{\infty} c_n z^n$ gives :

$$1 + \sum_{n=2}^{\infty} n c_n z^{n-1} = \sum_{m\geq 2} c_m z^{m-2}$$

So, $c_2 = 1$ (coeff of z^0) ; $c_3 = 2c_2 = 2$ (coeff of z^1) ; $c_4 = 3c_3 = 3!$ (coeff of z^2)

$\ldots$ $c_m = (m-1)!$ $\Rightarrow$ $\boxed{\varphi(z) = \sum_{n=1}^{\infty} (n-1)!\, z^n}$ divergent series.