Recap: linear, N-th order, homogeneous differential equation

$$f^{(N)}(z) = \sum_{j=1}^{N} a_j(z) f^{(N-j)}(z)$$ can be written in matrix form

$a_N \not\equiv 0$.

$$(*) \quad \boxed{\psi'(z) = A(z)\psi(z)} \quad \text{where} \quad \psi = \begin{bmatrix} f \\ f' \\ \vdots \\ f^{(N-1)} \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \\ a_N & \cdots & & & a_1 \end{bmatrix}$$

Companion matrix of $t^N - \sum_{j=1}^{N} a_j t^{N-j}$

Consider a system of the form $(*)$, where $A : \Omega \to \mathbb{C}$ is a hol. fn. An isolated singularity $\alpha \in \mathbb{C}$ of A (say, not essential) is said to be a regular point if α is a removable singularity

regular singular, or Fuchsian, if α is a simple pole (i.e., pole of order 1)

irregular (of Poincaré rank r) if α is a pole of order $r \geq 2$.

§1. Examples.– regular point – $A(z) = \sum_{n=0}^{\infty} A_n z^n$ near 0

$\psi'(z) = A(z)\psi(z)$ is solved by $\sum_{n=0}^{\infty} \psi_n z^n$ where

$$\boxed{(n+1)\psi_{n+1} = \sum_{j=0}^{n} A_j \psi_{n-j}}$$ $\forall\, n \geq 0$; ψ_0 can be arbitrary (initial condition).

Ex. Show that if $A(z)$ has non-zero radius of convergence, then so does $\psi(z)$ defined by these recursive formulae.

Constant coefficients $\psi'(z) = A\psi(z)$, $A \in M_{N\times N}(\mathbb{C})$ constant

is solved by e^{Az}. This method goes back to ~~Newton~~ Euler (?)

Solve: $y^{(N)} + a_{N-1}y^{(N-1)} + \dots + a_0 y = 0$. Set $y = e^{rt}$,

We get $\left(r^N + a_{N-1}r^{N-1} + \dots + a_0\right)e^{rt} = 0$. So, if $r_1, \dots, r_N$ are

distinct root of this polynomial (assuming such is the case), we get $\{e^{r_1 t}, \dots, e^{r_N t}\}$ a basis of

solutions. Note: distinct roots assumption $\Rightarrow$ A is diagonalizable

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \\ -a_0 & \cdots & & \cdots & -a_{N-1} \end{bmatrix}$$

with $A = G \begin{bmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_N \end{bmatrix} G^{-1}$. for some $G \in GL_N(\mathbb{C})$.

So, $e^{Az} = G \begin{bmatrix} e^{r_1 z} & & 0 \\ & \ddots & \\ 0 & & e^{r_N z} \end{bmatrix} G^{-1}$.

§2. Regular singular case.- Say, $A(z) = \dfrac{\Lambda}{z} + A_0 + A_1 z + \dots$ near $z=0$.

$\psi'(z) = A(z)\psi(z)$ - our solution near 0 is bound to be multivalued ($f' = \frac{a}{z}f$ is solved by $f = z^a = e^{a\log(z)}$ - multivalued)

Set $\psi(z) = H(z)\cdot z^{\Lambda}$.

Then $\Psi'(z) = H'(z)\, z^{\Lambda} + H(z) \cdot \frac{\Lambda}{z} z^{\Lambda}$

$$= \left(\frac{\Lambda}{z} + A_{reg}(z) \right) H(z) \cdot z^{\Lambda}.$$

So, our equation for ψ, takes the following form for H:

$$H'(z) = \cancel{\frac{1}{z}(H(z)\Lambda - \Lambda} = \frac{1}{z}\left(\Lambda H(z) - H(z)\Lambda\right) + A_{reg}(z) H(z).$$

Set $H(z) = H_0 + H_1 z + H_2 z^2 + \cdots$; $H_0 = Id_{N\times N}$. We get the following by comparing coefficients of z^n:

$$(n+1) H_{n+1} = (\Lambda H_{n+1} - H_{n+1}\Lambda) + \sum_{j=0}^{n} A_j H_{n-j}$$

The operator $X \mapsto \Lambda X - X\Lambda$ is usually written as $ad(\Lambda)$.

The recurrence relation becomes

$$\left(n+1 - ad(\Lambda)\right) H_{n+1} = \sum_{j=0}^{n} A_j H_{n-j}$$

Theorem. - (Frobenius) - Assume that eigenvalues of $ad(\Lambda)$ are not in $\mathbb{Z}_{\geq 1}$ (this condition is often called non-resonance condition).

Then $\sum_{n=0}^{\infty} H_n z^n$ is uniquely determined by $H_0 = Id_{N\times N}$

$$\left((n+1) - ad(\Lambda)\right) H_{n+1} = \sum_{j=0}^{n} A_j \cdot H_{n-j}$$

Moreover, if $A_{reg}(z)$ has non-zero radius of convergence, then so does $H(z)$.

Hint for a proof.- The first part is clear. By our assumption,

$$(n+1) - \mathrm{ad}(\Lambda) : M_{N\times N}(\mathbb{C}) \to M_{N\times N}(\mathbb{C}) \text{ is invertible.}$$

So $\exists!\ H_{n+1}$ such that $(n+1 - \mathrm{ad}(\Lambda))H_{n+1} = \sum_{j=0}^{n} A_j H_{n-j}$.

For the second part, we use Weierstrass' "majoring series" trick.

Assume radius of convergence of $A_{reg}(z) = 1$. Let $r \in (0,1)$;

and $M \in \mathbb{R}_{>0}$ be s.t. $\|A_n\| \cdot r^n \leq M$, $\forall n \geq 1$.

Choose $c \in \mathbb{R}_{>0}$ s.t. $\|((n+1) - \mathrm{ad}(\Lambda))^{-1}\| < \frac{c}{n+1}$ $\forall n \geq 0$.

Then $\sum_{n=0}^{\infty} H_n z^n$ is majorized by $\sum_{n=0}^{\infty} h_n z^n$ where $h_0 = 1$

$\left(\sum_{n=0}^{\infty} A_n z^n \text{ is majorized by } \frac{M}{1-r} \right)$ and $h_{n+1} = \frac{c}{(n+1)} \sum_{j=0}^{n} \frac{M}{r^j} h_{n-j}$

i.e., $h'(z) = \frac{cM}{1-rz} h(z)$

$$\Rightarrow h(z) = \exp\left(-\frac{cM}{r}\log(1-rz)\right) \text{ well-defined for } |z| < \frac{1}{r}.$$

□

§3. Irregular case (rank 2)

Assume $A(z) = \frac{\Lambda}{z^2} + \frac{X}{z} + A_0 + A_1 z + \dots$ near $z=0$.

(note: $f'(z) = \left(\frac{\lambda}{z^2} + \frac{x}{z}\right) f$ is solved by $e^{-\lambda/z} \cdot z^{x}$).

For simplicity, assume Λ is diagonal $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix}$

Write $X = \underset{\substack{\uparrow \\ \text{diagonal part}}}{X_d} + X_o$ ← off-diagonal part.

Following Frobenius, one can look for solutions of the form

$$\psi(z) = Y(z) \cdot e^{-\frac{\Lambda}{z}} \cdot z^{X_d}$$

$$\psi'(z) = Y'(z)\, e^{-\Lambda/z} z^{X_d} + \frac{1}{z^2} Y(z) \Lambda\, e^{-\Lambda/z} z^{X_d} + \frac{1}{z} Y(z)\, e^{-\Lambda/z} X_d\, z^{X_d}$$

$$= \left(\frac{1}{z^2} \Lambda + \frac{X}{z} + A_{reg}(z) \right) Y(z)\, e^{-\Lambda/z} z^{X_d}.$$

Note X_d and Λ commute. So, we get

$$Y'(z) = \left(\frac{\mathrm{ad}(\Lambda)}{z^2} + \frac{\mathrm{ad}(X_d)}{z} + \frac{X_0}{z} + A_{reg}(z) \right) Y(z). \quad - (**)$$

<u>Theorem.</u>– assuming $\lambda_1, \dots, \lambda_N$ are distinct.–

$\exists!\ Y(z) = \sum_{n=0}^{\infty} Y_n z^n$; $Y_0 = Id_{N \times N}$ satisfying this recurrence relation.

(<u>Ex.</u> Prove that $(**)$ determines $\begin{cases} \text{off-diagonal part of } Y_{n+2} \\ \text{\& diagonal part of } Y_{n+1} \end{cases}$ $\forall\, n \geq -1.$

assuming $Y_0, \dots, Y_n$ and off-diag. part of Y_{n+1} are known.

Argue by induction. $\square$)

<u>Serious issue</u>: This $Y(z)$ may have zero radius of convergence.

e.g. $\psi'(z) = \left(\frac{1}{z^2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{z}\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \right) \psi(z)$. $\quad (\lambda \in \mathbb{C})$

i.e. $\psi'(z) = \begin{bmatrix} 1/z^2 & \lambda/z \\ 0 & 0 \end{bmatrix} \psi(z)$. So, the second component of ψ is a constant

$\psi(z) = \begin{bmatrix} \alpha(z) \\ c \end{bmatrix}$. $\quad \alpha'(z) = \frac{\alpha(z)}{z^2} + \frac{\lambda c}{z}$.

Take $c = 0, 1$ to get $\begin{bmatrix} \alpha_1(z) & \alpha_2(z) \\ 0 & 1 \end{bmatrix}$

By our recipe $\psi(z) = Y(z)\, e^{-\frac{\Lambda}{z}} \cdot z^{X_d}$ $\quad \left(\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ X_d \equiv 0 \right)$

$$= Y(z) \begin{pmatrix} e^{-1/z} & 0 \\ 0 & 1 \end{pmatrix}$$

$\psi(z) = \begin{bmatrix} \alpha_1(z) & \alpha_2(z) \\ 0 & 1 \end{bmatrix}$ where $\alpha_1'(z) = \frac{\alpha_1(z)}{z^2}$; so $\alpha_1(z) = e^{-1/z}$

$$\alpha_2'(z) = \frac{\alpha_2(z)}{z^2} + \frac{\lambda}{z}$$

So, $Y(z) = \begin{bmatrix} 1 & \alpha_2(z) \\ 0 & 1 \end{bmatrix}$

$$\alpha_2(z) = -\lambda z - \lambda z^2 - 2\lambda z^3 - 6\lambda z^4 - \cdots$$

$$= -\lambda \cdot \sum_{n=1}^{\infty} (n-1)!\, z^n$$