§1. Asymptotic power series. –

Let S be a subset of $\mathbb{C}$; $z_0 \in \partial S$ (boundary of S)

f a holomorphic function so that

$S \subset$ domain of f.

We say $\displaystyle f \sim \sum_{n=0}^{\infty} a_n (z-z_0)^n$ as $z \to z_0$; $z \in S$ if

(to be read as: f is asymptotic to $\displaystyle\sum_{n=0}^{\infty} a_n (z-z_0)^n$)

for every $n \geq 0$, $\displaystyle \lim_{\substack{z \to z_0 \\ z \in S}} \frac{\left(f(z) - \sum_{m=0}^{n-1} a_m (z-z_0)^m \right)}{(z-z_0)^n}$ exists.

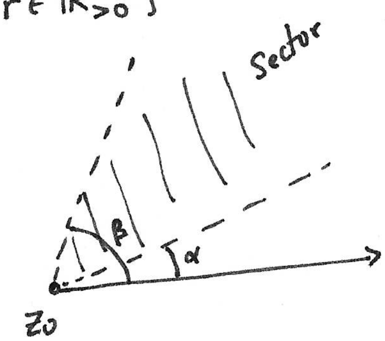
In practice S will be a sector of prescribed opening. Often $z_0 = \infty$.

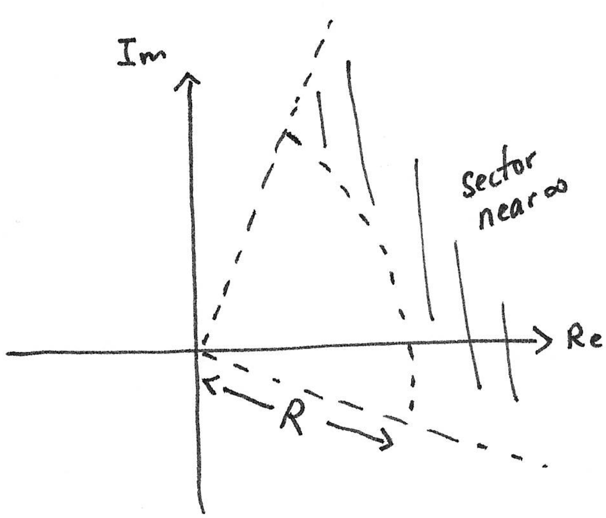
$$S(z_0; \alpha, \beta) = \{ z_0 + re^{i\theta} : \alpha < \theta < \beta ; r \in \mathbb{R}_{>0} \}$$

$\alpha < \beta$; $\beta - \alpha \leq 2\pi$.

or, near ∞,

$$S_R(\alpha, \beta) = \{ re^{i\theta} : \alpha < \cancel{\beta}\theta < \beta , r > R \}$$





For the effective use of asymptotic analysis, we are mostly interested in the case when z_0 is an essential singularity.

§2. Remarks. - (1) More general notions of asymptotic expansions also exist. For instance, one can start from a sequence of "approximating functions" $\{\phi_n(z)\}_{n=0}^{\infty}$

Defn. - $\{\phi_n : \mathcal{U} \to \mathbb{C}\}_{n=0}^{\infty}$; $\mathcal{U} \subset \mathbb{C}$ an open set containing S
holomorphic

is an approximating sequence if for each $n \geq 0$; $\lim\limits_{\substack{z \to z_0 \\ z \in S}} \dfrac{\phi_{n+1}(z)}{\phi_n(z)} = 0$

and $\lim\limits_{\substack{z \to z_0 \\ z \in S}} \phi_0(z)$ exists (say $= 1$ to fix ideas).

e.g. $\{1, z-z_0, (z-z_0)^2, \dots\}$ near $z_0 \in \mathbb{C}$.

$\left\{\dfrac{\sin(z)}{z}, \dfrac{\sin^2(z)}{z}, \dots\right\}$ near $z_0 = 0$.

$\{1, z^{-1}, z^{-2}, \dots\}$ near $z_0 = \infty$

examples of approximating sequences.

Non-example $\left\{\dfrac{1}{z}, \dfrac{1}{z+1}, \dfrac{1}{z+2}, \dots\right\}$ near $z = \infty$.

$\left(\lim\limits_{z \to \infty} \dfrac{z+n}{z+(n+1)} = 1 \text{ not } 0\right)$

$f \sim \sum\limits_{n=0}^{\infty} a_n \phi_n$ if for each $n \geq 0$,

(as $z \to z_0$, $z \in S$)

$$\lim_{\substack{z \to z_0 \\ z \in S}} \frac{f(z) - \sum\limits_{m=0}^{n-1} a_m \phi_m(z)}{\phi_n(z)} \text{ exists.}$$

(definition due to Poincaré (1896) - Sur les intégrales irregulières des équations linéaires.)

(2) Let $f(z) = e^{1/z}$; $z_0 = 0$; $\Omega = \mathbb{C} \setminus \{0\}$ domain of f.

$$\left| e^{1/z} \right| = e^{Re\left(\frac{1}{z}\right)} = e^{Re\left(\frac{\bar{z}}{|z|^2}\right)} = e^{x/x^2+y^2} \quad \text{if } z = x+iy.$$
$$= e^{\frac{\cos(\theta)}{r}} \quad \text{if } z = re^{i\theta}$$

As $z \to 0$, $Re(z) > 0$, $\left| e^{1/z} \right| \to +\infty$ faster than any poly.

(so $\arg(z) \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and cosine > 0) i.e., $\lim_{x \to 0^+} x^m e^{1/x} = \infty$. $\forall m \geq 0$

As $z \to 0$, $Re(z) < 0$, $\left| e^{1/z} \right| \to 0$ faster than any poly.

$e^{\frac{1}{z}} \sim 0$ as $z \to 0$, $Re(z) < 0$. $\leftarrow$ non-uniqueness - different functions can have the same asymptotic expansion.

(3) Example. Let $I(z) = \sum_{n=1}^{\infty} (-1)^{n-1} (n-1)!\, z^n$.

This formal power series was first studied by Euler who wanted to compute $\sum_{n=0}^{\infty} (-1)^n \cdot n! = I(1)$.

Hardy's book "Divergent series" has very detailed discussion of Euler's work and subsequent results about $I(z)$. (Chapter II, §B.)

The formula $I(z) = \sum_{n=1}^{\infty} (-1)^{n-1} (n-1)!\, z^n$ does not define a holomorphic function for us.

(we also encountered this series in the previous lecture, as a "formal solution" near an irregular singularity of Poincaré rank 2).

However, for z real and small enough, the value of the series can be computed, within an arbitrary error tolerance.

e.g. $z = \frac{1}{10}$; $\sum_{n=1}^{10} \frac{(-1)^{n-1} (n-1)!}{10^n}$ computes the real answer up to 3 decimal places.

$z = \frac{1}{100}$; Three terms give accuracy up to 5 decimal places.

$\rightarrow$ $z=1$, the case Euler was interested in, is "too big".

For small (real) values of z, a meaning can be assigned to the divergent series - in terms of error tolerance, as follows.

(4) Assume $f(z) \sim \sum_{k=1}^{\infty} a_k (z-z_0)^k$ as $z \to z_0$; $z \in S$.

Let $E > 0$ be arbitrary, and fixed (error tolerance).

By defn, for each $n \geq 0$, for each $\varepsilon > 0$

there exists $p(n, \varepsilon) > 0$ s.t.

$$\underset{z \in S}{|z| < p(n,\varepsilon)} \Rightarrow \left| f(z) - \sum_{k=0}^{n} a_k (z-z_0)^k \right| < \varepsilon \cdot (z-z_0)^n$$

Furthermore, assume that we are only interested in ~~$|z| \leq r$~~ $|z-z_0|<r$.

– for some fixed $r \in \mathbb{R}_{>0}$.

So, it suffices to consider $\varepsilon = \dfrac{E}{r^n}$.

Pick smallest N such that $\rho\left(N, \frac{E}{r^N}\right) > r$. (assume such an N exists – by shrinking r – if necessary)

Then, $$\left| \sum_{k=0}^{N} a_k (z-z_0)^k - f(z) \right| < E.$$
$$\forall\, z \in D(z_0; r) \cap S$$

§3. Lemma. If $f : D^\times(z_0; r) \to \mathbb{C}$ is holomorphic and has an asymptotic (power) series expansion to order 0, then f is holomorphic on $D(z_0; r)$ (i.e., z_0 is a removable singularity)

(note: $D^\times(z_0; r) = S(z_0; 0, 2\pi+\delta)$ for any $\delta > 0$.)

By asymptotic expansion to order N, we mean

$$\forall\, 0 \leq n \leq N, \quad \lim_{\substack{z \to z_0 \\ z \in S}} \left| \frac{f(z) - \sum_{k=0}^{n-1} a_k (z-z_0)^k}{(z-z_0)^n} \right| \text{ exists.}$$

Proof.- The hypothesis implies that $\lim_{z \to z_0} f(z)$ exists, hence z_0 is a removable singularity of f. □

Warning: In some textbooks (e.g. W. Wasow: Asymptotic expansions...) sectors of opening $> 2\pi$ are allowed; but, for instance, $S(0; 0, 3\pi) \neq D^\times(0; r)$

These sectors, let us denote them by $\tilde{S}(0;\alpha,\beta)$,

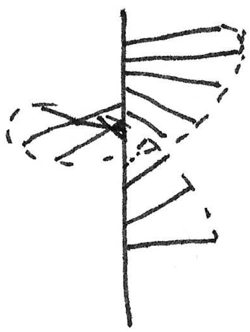
$\alpha < \beta$, $\left(\beta-\alpha \underset{>0}{\text{arbitrary}}\right)$.

are subsets of a Riemann surface

e.g. $\tilde{S}(0;\alpha,\beta) \subset \bigsqcup_{w\in \overset{\times}{D}(0;r)} \mathcal{O}_w$ (recall $\mathcal{O}_{z_0} \cong \mathbb{C}\{z-z_0\}$

neighbourhood of $(w,g)\in \mathcal{O}_w$; r_g = r.o.c. of g

$N(w,g;r) = \{(z,f) : z\in D(w,r_g),\ f=[g]_{\text{near } z}\}$)



"sector of opening $> 2\pi$"

§4. Origins of asymptotic series.

- Laurent series near an essential singularity
  - a divergent series.
- Solutions of differential equations near an irregular point.
- Solutions of difference equations.

We often try to find integral representation of some function, holomorphic on a sector, with given asymptotic expansion.
Techniques for finding such functions are often called "resummation methods"

Example. $I(z) = \sum_{n=0}^{\infty} (-1)^{n-1} (n-1)!\, z^n$ formally solves

$$I'(z) = \frac{z - I(z)}{z^2} \quad ; \quad \text{i.e.} \quad \boxed{z^2 I'(z) + I(z) - z = 0}$$

Assume $\mathrm{Re}(z) > 0$. Then $z = \int_0^\infty e^{-p/z}\, dp$.

Postulate - our solution will be of the form $\int_0^\infty \varphi(p)\, e^{-p/z}\, dp =: F(z)$

$$F'(z) = \int_0^\infty \varphi(p)\, e^{-p/z} \cdot \frac{p}{z^2}\, dp$$

(Laplace transform of φ).

(if differentiation under the integral sign is permissible).

$$z^2 F' + F - z = \int_0^\infty \big((p+1)\varphi(p) - 1\big)\, e^{-p/z}\, dp = 0 \quad \text{if}$$

$$\varphi(p) = \frac{1}{1+p}. \qquad \text{Hence} \quad F(z) := \int_0^\infty \frac{e^{-p/z}}{1+p}\, dp$$

$z \in \mathbb{H}_{\text{right}}$ ($\mathrm{Re}(z) > 0$) - right half-plane

Once we have checked uniform convergence of this infinite integral, we will be able to say $F : \mathbb{H}_{\text{right}} \to \mathbb{C}$ is a holomorphic solution of $z^2 F' + F - z = 0$.