Lecture 12

Differential equations near irregular singularity $\rightsquigarrow$ (power series) Asymptotic expansions*

$$\left[f(z) \sim \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{as } z \to z_0 \,;\ z \in S \right]$$

Difference equations (example below) $\rightsquigarrow$ means: $\forall n \geq 0$,

$$\lim_{\substack{z \to z_0 \\ z \in S}} \frac{f(z) - \sum_{k=0}^{n-1} a_k (z-z_0)^k}{(z-z_0)^n} \text{ exists.}$$

Summability of $\sum_{n=0}^{\infty} c_n$ $\rightsquigarrow$

(Asymptotic expansions capture (computational) behaviour of $f(z)$ near an essential singularity z_0, as z varies over a sectorial neighbourhood S of z_0.)

§1. A difference equation.

$$\boxed{f(z-1) - f(z) = \frac{1}{z^2}}$$

Easy exercise - show that this equation has no rational solutions.

Soln. 1:

$$f(z) = \frac{-1}{z^2} + f(z-1)$$

$$= \dots = -\sum_{n \geq 0} \frac{1}{(z-n)^2}$$

Q: Is this convergent?

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* This definition first appeared in Poincaré (1896) - Henri Poincaré (1854-1912).

Soln 2. Shift z by 1 to get $f(z) - f(z+1) = \frac{1}{(z+1)^2}$

So, $$f(z) = \frac{1}{(z+1)^2} + f(z+1)$$
$$= \cdots = \sum_{n \geq 1} \frac{1}{(z+n)^2}.$$

Soln 3 - Formal solution - There is a unique formal power series $\sum_{n=0}^{\infty} c_n z^{-n-1} \in \mathbb{C}[[z^{-1}]]$ s.t. $F(z-1) - F(z) = \frac{1}{z^2}$ (say, $F(z)$)

(This series is necessarily divergent. Since, on the contrary, if $F(z)$ converges (uniformly) for $\{|z| > R\}$, then we can extend it to a mero fn. $\mathbb{C} \dashrightarrow \mathbb{C}$ via the equation $F(z-1) = \frac{1}{z^2} + F(z)$. Within $\{|z| \leq R\}$, it can have only finitely many poles; and $F(\infty) = 0$. So, $F(z)$ must be Taylor series expansion of a rational function solving $f(z-1) - f(z) = z^{-2}$. But there are no rational solns. to this equation.)

Coefficients $\{c_n\}_{n=0}^{\infty}$: $$F(z-1) = \sum_{n=0}^{\infty} c_n (z-1)^{-n-1}$$
$$= \sum_{n=0}^{\infty} c_n \cancel{(-1)^{n+1}} z^{-n-1} (1 - z^{-1})^{-n-1}$$
$$= \sum_{n=0}^{\infty} c_n (+z)^{-n-1} \cdot \left(\sum_{l \geq 0} \binom{n+l}{l} z^{-l} \right) = \sum_{N \geq 0} z^{-N-1} \cdot \left(\sum_{K=0}^{N} \cancel{(-1)^{K+1}} \binom{N}{K} c_K \right)$$

$$F(z-1) - F(z) = z^{-2} \Rightarrow \sum_{N=0}^{\infty} z^{-N-1}\left(\sum_{K=0}^{N-1} \binom{N}{K} c_K\right) = \frac{1}{z^2}.$$

(so $N \geq 1$.)

z^{-2} : $c_0 = 1$. z^{-3} : $c_0 + 2c_1 = 0 \Rightarrow c_1 = \frac{-1}{2}$.

z^{-4} : $c_0 + 3c_1 + 3c_2 = 0 \Rightarrow 3c_2 = -1 + \frac{3}{2} \Rightarrow c_2 = \frac{1}{6}$.

z^{-5} : $c_0 + 4c_1 + 6c_2 + 4c_3 = 0 \Rightarrow 4c_3 = -1 + 4\left(\frac{1}{2}\right) - 6\left(\frac{1}{6}\right) = 0$

So, $c_3 = 0$.

$\ldots$ z^{-n-2} : $\sum_{K=0}^{n} \binom{n+1}{K} c_K = 0$; So $\boxed{(n+1)\, c_n = -\sum_{K=0}^{n-1} \binom{n+1}{K} c_K}$.

Compare with Bernoulli numbers, defined by (see Problem 19)

$$\beta(t) := \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

$\to$ we get that $c_n = B_n$ $(\forall\, n \geq 0)$.

Derivation : $z^{-2} = \int_0^{\infty} p \cdot e^{-pz}\, dp$ for $Re(z) > 0$

Assume our solution is to be of the form

$F(z) = \int_0^{\infty} \varphi(p)\, e^{-pz}\, dp$. $F(z-1) - F(z) = \int_0^{\infty} (e^p - 1)\varphi(p) e^{-pz}\, dp$

So, $\varphi(p) = \dfrac{p}{e^p - 1} = \beta(p)$ (Bernoulli number's exp-generating function).

Formally (this is a valid computation of asymptotic series expansion of a Laplace integral - a result known as Watson's lemma - to be proved later).

$$F(z) = \int_0^\infty \frac{p}{e^p - 1} e^{-pz}\, dp = \int_0^\infty \left(\sum_{n=0}^\infty B_n \frac{t^n}{n!} \right) e^{-tz}\, dt$$

Exercise $\displaystyle\int_0^\infty \frac{t^n}{n!} e^{-tz}\, dt = z^{-n-1}$ for $Re(z) > 0$.

So $\displaystyle F(z) \sim \sum_{n=0}^\infty B_n z^{-n-1}$. (Exercise - prove directly that $\displaystyle\sum_{n=0}^\infty B_n z^{-n-1}$ has 0 radius of convergence).

§2. Summation of series - some historical remarks. -

Leibniz (1646-1716) - in a letter to Johann Bernoulli, dated 1696, states $1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$ on grounds of "probability".

i.e. $\left\{ \underbrace{1 - 1 + 1 - 1 \cdots}_{n\text{-terms}} = \begin{matrix} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{matrix} \right\}_{n=1}^{\infty}$ has 0 & 1 with same frequency $= \frac{1}{2}$.

Euler (1707 - 1783), to N. Bernoulli (1743):

(1) if $s = 1-1+1-1+\cdots$ then $s = 1-s$, so, $s = \frac{1}{2}$.

(2) $1 + x + x^2 + \cdots = \frac{1}{1-x}$. Set $x=-1$ to get $1-1+1-1\cdots = \frac{1}{2}$.

(3) (Euler transform) Assume we want to compute $\sum_{n=0}^{\infty} a_n$.

Define $a_n(q) = \frac{1}{(q+1)^{n+1}} \sum_{\ell=0}^{n} \binom{n}{\ell} q^{\ell} a_{n-\ell}$ and

compute $\sum_{n=0}^{\infty} a_n(q)$.

In our case, $a_n = (-\alpha)^n$ and $\sum_{n=0}^{\infty} a_n = \frac{1}{1+\alpha}$ for $|\alpha| < 1$.

So, $a_n(q) = \frac{1}{(q+1)^{n+1}} \sum_{\ell=0}^{n} \binom{n}{\ell} q^{\ell} (-\alpha)^{n-\ell} = \frac{(q-\alpha)^n}{(q+1)^{n+1}}$.

$$\sum_{n=0}^{\infty} a_n(q) = \frac{1}{q+1}\left(1 - \frac{q-\alpha}{q+1}\right)^{-1} \quad \text{for } |q-\alpha| < |q+1|$$

$$= \frac{1}{1+\alpha}$$ now it is permitted to set $\alpha = 1$ and get $\frac{1}{2}$.