Lecture 13

§1. Functions defined by (infinite) integrals.

Assume $\Omega \subset \mathbb{C}$ open, connected set and $G(z,t) : \Omega \times \mathbb{R}_{>0} \to \mathbb{C}$ are given. Furthermore, assume that G satisfies the following hypotheses.

(1) G is continuous on $\Omega \times \mathbb{R}_{>0}$.

(2) For a fixed $t \in \mathbb{R}_{>0}$, $G(t, -) : \Omega \to \mathbb{C}$ is holomorphic

(3) $\partial_z G : \Omega \times \mathbb{R}_{>0} \to \mathbb{C}$ is again continuous.

(4) $\displaystyle \lim_{\substack{\varepsilon \to 0^+ \\ R \to \infty}} \int_\varepsilon^R G(z,t)\, dt$ converges uniformly (rel. to cpct subsets of Ω).

<u>Theorem</u>. - $\displaystyle g(z) := \int_0^\infty G(z,t)\, dt$ is a holomorphic function of z and

$$g'(z) = \int_0^\infty \partial_z G(z,t)\, dt.$$

<u>Proof</u>.- Expanding condition (4): pick $a_1 \geq a_2 \geq \cdots$ $\lim a_n = 0$

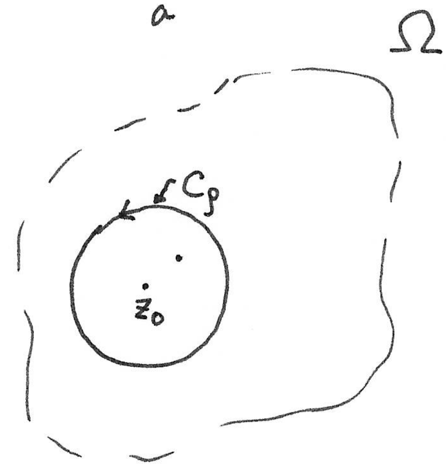
$b_1 \leq b_2 \leq \cdots$ $\lim b_n = \infty$

and define $\displaystyle g_n(z) = \int_{a_n}^{b_n} G(z,t)\, dt.$

Then, by (4), $\{g_n(z)\}_{n=1}^{\infty}$ is a uniformly convergent seq. of functions. So, to prove the theorem, it is enough to prove that each g_n is holomorphic and $g_n'(z) = \int_{a_n}^{b_n} \partial_z G(z,t)\,dt$.

(we are using Weierstrass' theorem on uniform convergence).

So, let $[a,b] \subset \mathbb{R}_{>0}$ be fixed and let $h(z) := \int_a^b G(z,t)\,dt$.



Let $z_0 \in \Omega$ and $\rho > 0$ be such that $\overline{D(z_0,\rho)} \subset \Omega$.

As for each t, $G(-,t)$ is holomorphic (assumption (2))

we get
$$h(z_0) = \int_a^b G(z_0,t)\,dt = \frac{1}{2\pi i}\int_a^b \int_{C_\rho} \frac{G(z,t)}{z-z_0}\,dz\,dt$$

(the remainder is same as our proof of Cauchy's formula)

<u>Proof that h is continuous.</u> - Let $\alpha \in D(0,\rho)$ (i.e., $|\alpha| < \rho$).

Then
$$h(z_0+\alpha) - h(z_0) = \frac{1}{2\pi i}\int_a^b \int_{C_\rho} G(z,t)\left(\frac{1}{z-z_0-\alpha} - \frac{1}{z-z_0}\right) dz\,dt$$

$$= \frac{\alpha}{2\pi i}\int_a^b \int_{C_\rho} \frac{G(z,t)}{(z-z_0)(z-z_0-\alpha)}\,dz\,dt$$

By triangle inequality,

$$\left| h(z_0+\alpha) - h(z_0) \right| \leq \frac{M \, |\alpha|}{2\pi \rho (\rho - |\alpha|)} \, 2\pi\rho \, (b-a) \to 0 \text{ as } |\alpha| \to 0.$$

$$M = \operatorname{Max} \left\{ |G(w,s)| : \begin{array}{l} w \in C_\rho \\ s \in [a,b] \end{array} \right\}.$$

□

Proof that h is holomorphic and $h'(z) = \int_a^b \partial_z G(z,t)\, dt$:

(same notations as above)

$$\frac{h(z_0+\alpha) - h(z_0)}{\alpha} = \frac{1}{2\pi i} \int_a^b \int_{C_\rho} \frac{G(z,t)}{(z-z_0-\alpha)(z-z_0)} \, dz \, dt$$

Claim (easy exercise) - $\displaystyle \lim_{\alpha \to 0} \int_a^b \int_{C_\rho} \frac{G(z,t)}{(z-z_0-\alpha)(z-z_0)} \, dz \, dt = \int_a^b \int_{C_\rho} \frac{G(z,t)}{(z-z_0)^2} \, dz \, dt$

So $\displaystyle h'(z_0) = \lim_{\alpha \to 0} \frac{h(z_0+\alpha) - h(z_0)}{\alpha}$ exists and is equal to

$$= \frac{1}{2\pi i} \int_a^b \int_{C_\rho} \frac{G(z,t)}{(z-z_0)^2} \, dz \, dt = \int_a^b \partial_z G(z_0, t) \, dt$$

Cauchy's integral formula

□

§2. Laplace transform. - Let $f(t)$ be a continuous function of $t \in \mathbb{R}_{>0}$.

Define $(\mathcal{L}f)(z) := \displaystyle\int_0^\infty f(t)\, e^{-zt}\, dt$. (Laplace transform of f).

Lemma. - Assume that :

(i) f has at most exponential growth as $t \to \infty$.

Meaning, $\exists\ M, C, R \in \mathbb{R}_{>0}$ s.t.

$$|f(t)| < M \cdot e^{Ct}, \quad \forall\, t > R$$

(ii) f has at worst logarithmic singularity as $t \to 0^+$, i.e.

$\exists\ a, b, r > 0$ s.t. $|f(t)| < b \cdot t^{a-1}$ $\forall\, t \in (0, r)$.

Then $(\mathcal{L}f)(z)$ is a holomorphic function of $z \in \{w : Re(w) > C\}$

$\Omega = \{ Re > C \}$. (same C as in (i))

Proof. We need to verify the hypotheses of Thm §1 above. The only non-trivial one is (4) (uniform convergence), which can be explicitly written as follows.

(To prove) Given $\varepsilon > 0$, there exist $r_1, R_1 > 0$ s.t. and $K \subset \Omega$ compact

$$\left| \int_0^s f(t)\, e^{-zt}\, dt \right| < \varepsilon, \quad \text{for every } s < r_1 \text{ and } z \in K. \quad (\text{near } 0)$$

$$\left| \int_S^\infty f(t)\, e^{-zt}\, dt \right| < \varepsilon \quad , \text{ for every } S > R_1 \ , \ z \in K.$$

(near ∞)

<u>Proof of the existence of r_1</u> (near 0) : Choose $r_1 < r$ (r as in hypothesis (2) on f).

such that $\quad r_1^{\,a} < \varepsilon \cdot \dfrac{a}{b}$.

Then, $\forall\, s < r_1$ we have

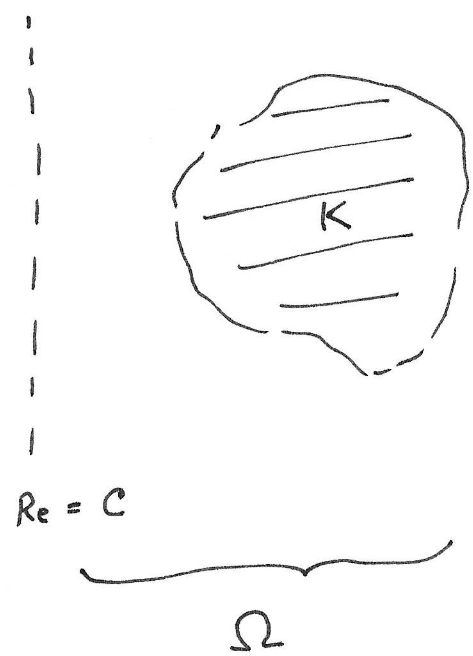
$$\left| \int_0^s f(t)\, e^{-zt}\, dt \right| < \int_0^s b \cdot t^{a-1}\, dt = b\, \frac{s^a}{a} < \varepsilon .$$

<u>Proof of the existence of R_1</u> (near ∞) :

Choose $A > C$ s.t. $\operatorname{Re}(z) > A \quad \forall z \in K.$

Let $R_1 > R$ be such that

$$e^{-(A-C)R_1} < \frac{\varepsilon\,(A-C)}{M} .$$



Then, $\forall\, S > R_1$ we have

$$\left| \int_S^\infty f(t)\, e^{-zt}\, dt \right| \leq \int_S^\infty |f(t)|\, e^{-\operatorname{Re}(z)t}\, dt$$

$$< M \int_S^\infty e^{-(A-C)t}\, dt = \frac{M e^{-(A-C)S}}{A-C} < \varepsilon \qquad \square$$

§3. Watson's lemma.– Again, let $f(t)$ be a continuous function of $t \in \mathbb{R}_{>0}$ and assume the hypotheses of Lemma §2 above hold.

Lemma. If $f(t) \sim \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$ as $t \to 0^+$ $(t \in \mathbb{R})$

then $(\mathcal{L}f)(z) \sim \sum_{n=0}^{\infty} a_n z^{-n-1}$ as $Re(z) \to \infty$.

Proof.– Let $N \geq 0$. We have to show that

$$\lim_{Re(z) \to \infty} z^N \left(\mathcal{L}f(z) - \sum_{n=0}^{N-1} a_n z^{-n-1} \right) = 0.$$

Note
$$\sum_{n=0}^{N-1} a_n z^{-n-1} = a_0 z^{-1} + \dots + a_{N-1} z^{-N} = \int_0^{\infty} \left(\sum_{n=0}^{N-1} a_n \frac{t^n}{n!} \right) e^{-zt}\, dt$$

Hence
$$(\mathcal{L}f)(z) - \sum_{n=0}^{\infty} a_n z^{-n-1} = \int_0^{\infty} f_N(t)\, e^{-zt}\, dt \quad ; \text{ where}$$

$$f_N(t) \sim \sum_{n=N}^{\infty} a_n \frac{t^n}{n!} \; ; \text{ as } t \to 0^+.$$

$$f_N(t) = f(t) - \sum_{n=0}^{N-1} a_n \frac{t^n}{n!}.$$

So, $\lim_{t \to 0^+} \frac{f_N(t)}{t^N} = \frac{a_N}{N!}$ exists.

This means that we can find $A, r > 0$ s.t.

$$|f_N(t)| < A \cdot t^N, \quad \forall\, t \in (0, r).$$

Let $M, R, C > 0$ be constants so that $|f_N(t)| < M \cdot e^{ct}$ $\forall\, t > R$.

(Note - f has at most exponential growth as $t \to \infty$ and $f_N = f -$ polynomial $\Rightarrow$ f_N has at most exponential growth.)

Write
$$\int_0^\infty f_N(t)\, e^{-zt}\, dt = \underbrace{\int_0^r}_{1} + \underbrace{\int_r^R}_{2} + \underbrace{\int_R^\infty}_{3} f_N(t)\, e^{-zt}\, dt$$

<u>Claim 1.</u> 2 and 3 are asymptotically 0 as $Re(z) \to \infty$.

<u>Proof</u> - 2 :
$$\left| \int_r^R f_N(t)\, e^{-zt}\, dt \right| \leq m \cdot \int_r^R e^{-xt}\, dt \qquad (x = Re(z))$$

$(m = \text{MAX} \{ |f_N(t)| : t \in [r, R] \})$.

$$= m \, \frac{e^{-rx} - e^{-Rx}}{x} \to 0 \text{ as } x \to \infty \text{ faster than any } x^N.$$

So
$$\lim_{Re(z) \to \infty} z^N \cdot \int_r^R f_N(t)\, e^{-zt}\, dt = 0.$$

3 :
$$\left| \int_R^\infty f_N(t)\, e^{-zt}\, dt \right| \leq M \int_R^\infty e^{(c-x)t}\, dt = M \cdot \frac{e^{-(x-c)R}}{x - c} \to 0 \text{ as } x \to \infty$$

$\square$

Finally, for ① $= \displaystyle\int_0^r f_N(t)\, e^{-zt}\, dt$:

$$\left| \int_0^r f_N(t)\, e^{-zt}\, dt \right| \leq A \int_0^r t^N e^{-xt}\, dt < A \cdot \int_0^\infty t^N e^{-xt}\, dt$$

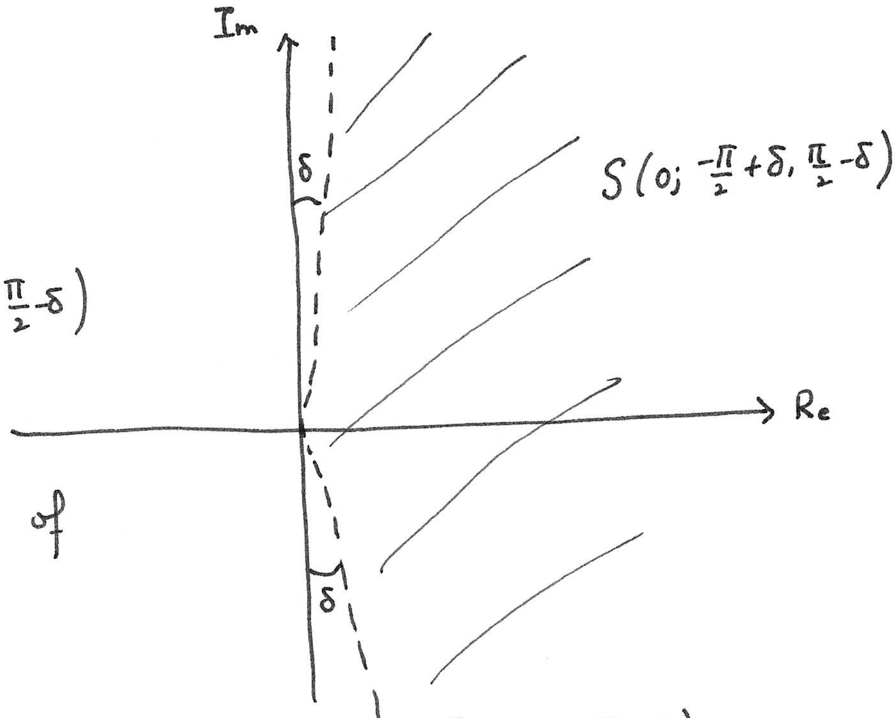
$$= A \cdot \frac{N!}{x^{N+1}}$$

So,

$$\lim_{\mathrm{Re}(z)\to\infty} \left| z^N \left((\mathcal{L}f)(z) - \sum_{n=0}^{N-1} a_n z^{-n-1} \right) \right| \leq \lim_{\mathrm{Re}(z)\to\infty} \frac{A \cdot N!\, |z|^N}{\mathrm{Re}(z)^{N+1}} = 0$$

as claimed. □

<u>Remark</u> If $z \in S(0; -\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta)$ $(\delta > 0$ small$)$.



then

$|z| \sin\delta \leq \mathrm{Re}(z) \leq |z|$.

So $z \to \infty$, $z \in S(-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta)$

$\Longleftrightarrow$ $\mathrm{Re}(z) \to \infty$.

We could write the conclusion of Watson's lemma as

$$\mathcal{L}f(z) \sim \sum_{n=0}^{\infty} a_n z^{-n-1} \quad \text{as} \quad z \to \infty; \ \arg(z) \in \left(-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta\right).$$