Lecture 14

Recall that last time we discussed properties of general integral transforms, introduced Laplace transform and proved Watson's lemma.

$$\boxed{\text{Kernel of Laplace transform} = e^{-zt}}$$

Input

$f(t)$: cnts fn. of $t \in \mathbb{R}_{>0}$ s.t.

- subexponential growth at ∞
- at worst logarithmic singularity as $t \to 0$

$\rightsquigarrow$

Output

$$F(z) = \mathcal{L}f(z) = \int_0^\infty f(t)\, e^{-zt}\, dt$$

holomorphic on $\mathrm{Re}(z) \gg 0$.

$F(z) \to 0$ as $\mathrm{Re}(z) \to \infty$.

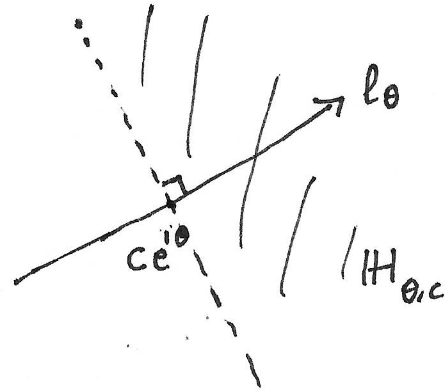
Watson's lemma. If $f(t) \sim \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$, as $t \to 0^+$, then

$$\mathcal{L}f(z) \sim \sum_{n=0}^{\infty} a_n z^{-n-1} \quad \text{as} \quad z \to \infty, \quad \arg(z) \in \left(-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta\right).$$

§1. Laplace transform along arbitrary ray.

Notations. for $\theta \in \mathbb{R}$, let $\ell_\theta = \{re^{i\theta} : r \in \mathbb{R}_{>0}\}$ be the ray emanating from 0 at an angle θ.

$\mathbb{H}_{\theta,c}$ = half plane defined by
$(c>0)$
$\{z \in \mathbb{C} : \mathrm{Re}(z e^{-i\theta}) > c\}$



Without much difficulty, the results stated on the previous page generalize to the following.

Theorem. Assume $f(t)$ is a $\mathbb{C}$-valued, cnts. function of $t \in \ell_{-\theta}$; satisfying:

(a) $|f(t)|$ has subexponential growth as $t \to \infty$, $t \in \ell_{-\theta}$.
i.e., $\exists\ M, C, R > 0$ s.t.
$|f(t)| < Me^{C|t|}$ for $|t| > R$, $t \in \ell_{-\theta}$.

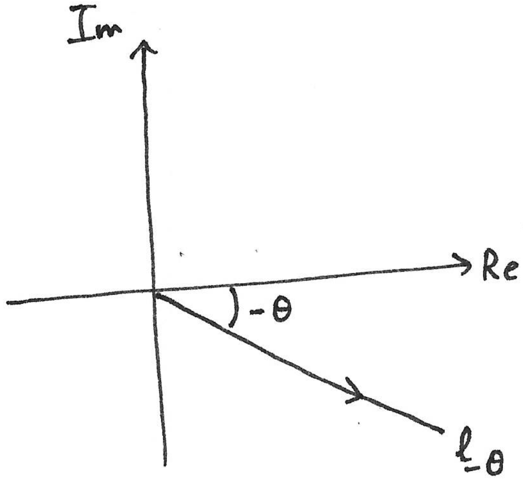
(b) $f(t)$ has at worst log-singularity as $t \to 0^+$, $t \in \ell_{-\theta}$ i.e.
$\exists\ a, b, r > 0$ s.t. $|f(t)| < b\,t^{a-1}$ for $|t| \in (0, r)$, $t \in \ell_{-\theta}$.

Then, $(\mathcal{L}_\theta f)(z) := \displaystyle\int_{\ell_{-\theta}} f(t)\, e^{-zt}\, dt$ defines a holomorphic function of z in $\mathbb{H}_{\theta, C}$. Moreover, if $f(t) \sim \displaystyle\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$ as $t \to 0^+$ along $\ell_{-\theta}$

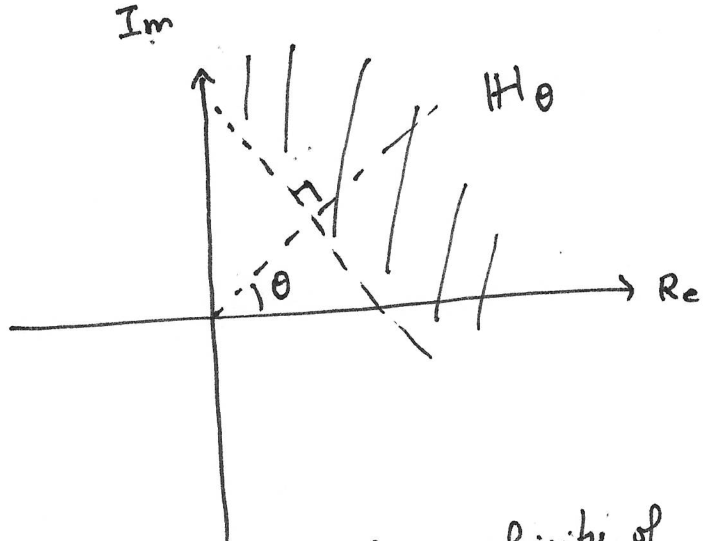
then $$(\mathcal{L}_\theta f)(z) \sim \sum_{n=0}^{\infty} a_n z^{-n-1} \quad \text{as} \quad \mathrm{Re}(z e^{-i\theta}) \to \infty.$$

This asymptotic expansion remains valid as $z \to \infty$ in Σ

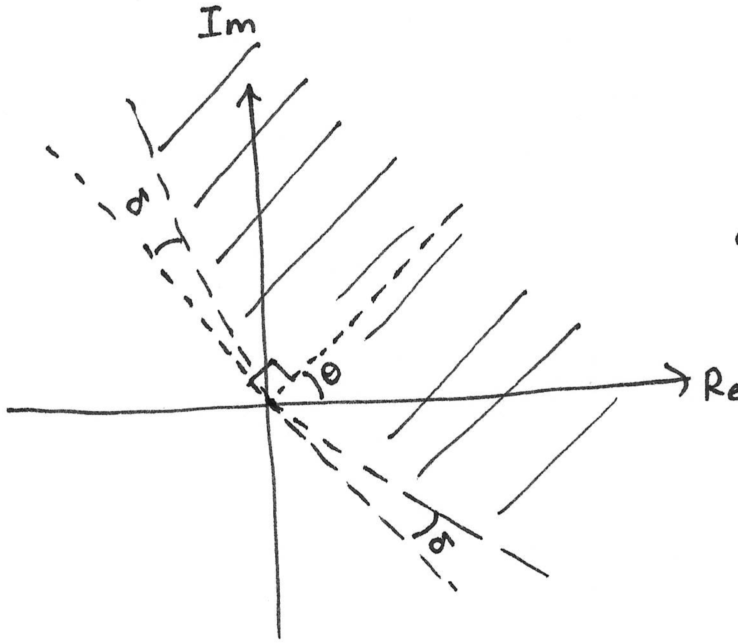
$$\Sigma = \left\{ re^{i\phi} : r > 0, \ \phi \in \left(\theta - \tfrac{\pi}{2} + \delta, \ \theta + \tfrac{\pi}{2} - \delta\right) \right\}.$$



line of integration for $\mathcal{L}_\theta$



Domain of holomorphicity of $\mathcal{L}_\theta f$.



Sector Σ of validity of the asymptotic expansion of $\mathcal{L}_\theta f$.

§2. Laplace transform to analytic continuation.

Assume $f(t)$ is a meromorphic function of $t \in \mathbb{C}$, with a discrete set $A \subset \mathbb{C}$ of poles.

Further, assume that $0 \notin A$ and $|f(t)| < M e^{c \cdot |t|}$ $\forall\, |t| > R$.

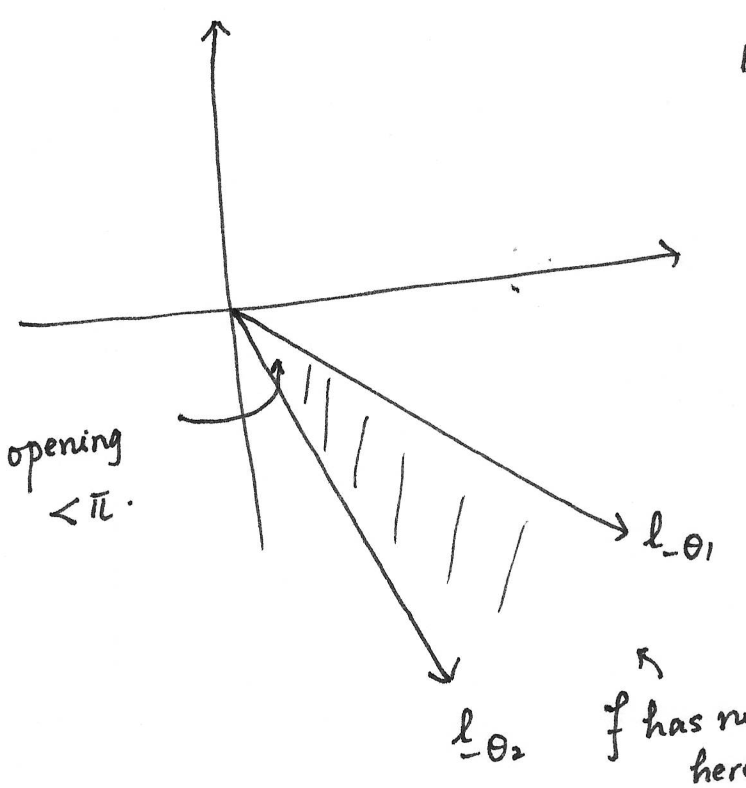
(f has sub-exponential growth along all rays.)

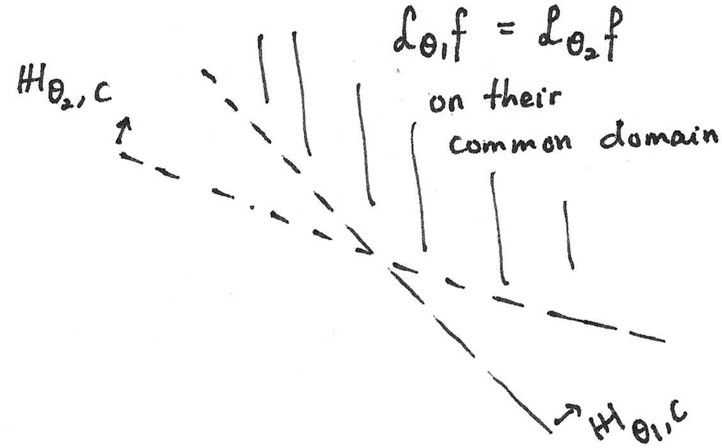
By our previous theorem, $(\mathcal{L}_\theta f)(z)$ is defined $\forall\, \theta$ s.t. $A \cap \ell_{-\theta} = \phi$. (rays should not pass through poles of f).

Lemma. If $\theta_1 < \theta_2$ are such that $\theta_2 - \theta_1 < \pi$

and $A \cap S(0; -\theta_2, -\theta_1) = \phi$ (no poles in between $\ell_{-\theta_1}$ and $\ell_{-\theta_2}$)

Then $\mathcal{L}_{\theta_1} f = \mathcal{L}_{\theta_2} f$ $\forall\, z \in \mathbb{H}_{\theta_1, c} \cap \mathbb{H}_{\theta_2, c}$.





<u>Proof.</u> Pick $\rho > R$ and $z \in \mathbb{H}_{\theta_1, C} \cap \mathbb{H}_{\theta_2, C}$

$C_1 > C$ such that $\operatorname{Re}(z e^{-i\phi}) > C_1 (> C)$.

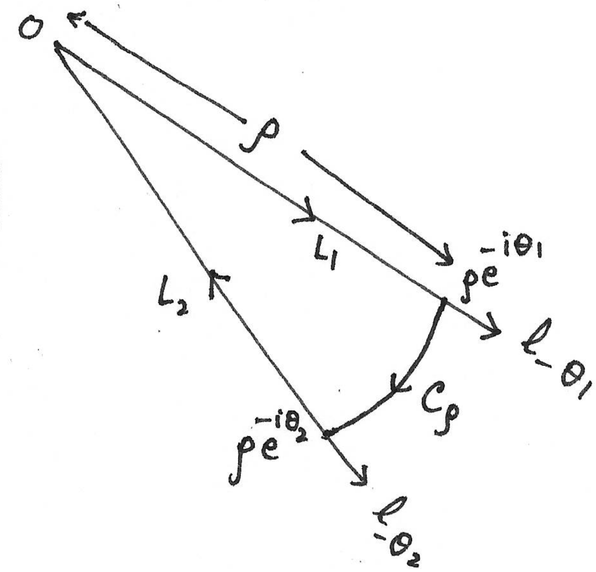
$\forall\, \phi \in (\theta_1, \theta_2)$

(if $\theta_2 - \theta_1 > \pi$, this cannot be done.)

$$\mathcal{L}_{\theta_1} f - \mathcal{L}_{\theta_2} f$$

$$= \lim_{\rho \to \infty} \left(\int_{L_1 + L_2 + C_\rho} f(t) e^{-zt} dt - \int_{C_\rho} f(t) e^{-zt} dt \right)$$

$$= - \lim_{\rho \to \infty} \int_{C_\rho} f(t) e^{-zt} dt$$



Contour $L_1 \cdot C_\rho \cdot L_2$

Since $\displaystyle\int_{L_1 + L_2 + C_\rho} = 0$ by Cauchy's theorem.

Now
$$\left| \int_{C_\rho} f(t) e^{-zt} dt \right| < M e^{C \cdot \rho} \cdot e^{-C_1 \rho} \cdot (\theta_2 - \theta_1) \rho$$
$$= (\text{Constant}) \cdot \rho \cdot e^{-(C_1 - C)\rho} \to 0 \text{ as } \rho \to \infty \quad \square$$

$$\left(\left| e^{-zt} \right| = e^{-\operatorname{Re}(zt)} \; ; \; \operatorname{Re}(zt) = \operatorname{Re}(z \cdot \rho \cdot e^{-i\phi}) > \rho \cdot C_1 \right)$$
$$\phi \in (\theta_1, \theta_2)$$