Lecture 15

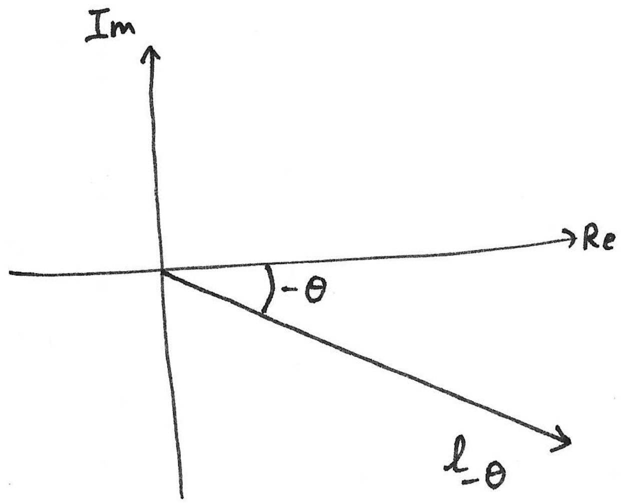
Recall: last time we were studying Laplace transform along arbitrary rays and relation among functions obtained this way.

Notations. - $\ell_\theta = \mathbb{R}_{>0} \cdot e^{i\theta}$

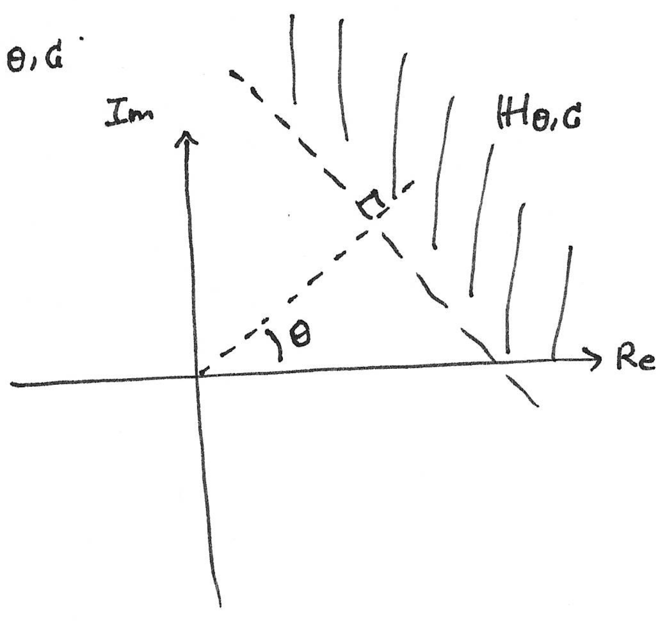
$\mathbb{H}_{\theta, C} = \{ z \in \mathbb{C} : \operatorname{Re}(z e^{-i\theta}) > C \}$ $\quad (\theta \in \mathbb{R}, C \in \mathbb{R})$

$$(\mathcal{L}_\theta f)(z) = \int_{\ell_{-\theta}} f(t)\, e^{-zt}\, dt \qquad \text{Laplace transform}$$

Under suitable hypotheses (see §1 below) on f, $\mathcal{L}_\theta f\,(z)$ is a holomorphic function on a half plane of the form $\mathbb{H}_{\theta, C}$.



t - plane



z - plane

$\mathcal{L}_\theta f$ holomorphic on $\mathbb{H}_{\theta, C}$

§1. Hypotheses on f. – We are assuming that $f : \mathbb{C} \dashrightarrow \mathbb{C}$ is a meromorphic function.

Let $A \subset \mathbb{C}$ be the (discrete) set of poles of f.

Assume $0 \notin A$ and it is possible to choose

$$R_1 < R_2 < \ldots \qquad \lim_{m \to \infty} R_m = \infty$$

in such a manner that

(i) $A \cap C(0; R_m) = \phi$ (f has no poles on the circle of radius R_m centered at 0).

(ii) $\exists$ constants $M, C > 0$ such that

$$|f(t)| < M e^{C R_m} \qquad \forall\, t \in C(0; R_m)$$

(f has sub-exponential growth as $|t| \to \infty$ along $R_1, R_2 \cdots$)

§2. Remarks – (1) If f has only finitely many poles, then we could have put simpler assumption of sub-exponential growth along any direction.

To incorporate the case of infinitely many poles, we are assuming that it is possible to stay a finite distance away from them and bound f by an exponential function.

(2) Similar assumptions, as made above, appeared in the work of Mittag-Leffler*, around 1880, who proved the following "generalized partial fractions" theorem. It is a more concrete version of what is known as Mittag-Leffler theorem.

Generalized partial fractions. Let $p(z)$ be a meromorphic function,

B = set of poles of p, $0 \notin B$ such that

(1) All the poles are of order 1 (simple poles). Let

$$c_K = \text{Residue of } p \text{ at } z = b_K .$$

(say $B = \{b_1, b_2, \dots\}$ arranged so as to have
$$0 < |b_1| \leq |b_2| \leq \dots \text{)}$$

(2) It is possible to choose $R_1 < R_2 < \dots$ $\lim_{m\to\infty} R_m = \infty$,

in such a manner that $B \cap C(0; R_m) = \phi$ and there is a constant $M > 0$ so that

$$|p(z)| < M \quad \left(\begin{array}{l}\forall z \text{ s.t. } |z| = R_m \\ \forall m\end{array}\right)$$

should work for every $m = 1, 2, \dots$

Then
$$\boxed{p(z) = p(0) + \sum_{K=1}^{\infty}\left(\frac{c_K}{z - b_K} + \frac{c_K}{b_K}\right)}.$$

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* Gösta Mittag-Leffler 1846 - 1927

<u>Mittag-Leffler Theorem.</u> Assume that a discrete set $B \subset \mathbb{C}$, $0 \notin B$ is given, and for each $n \geq 1$, we have $B = \{b_1, b_2, \dots\}$ a polynomial $P_n(z)$.

Then there exists a meromorphic function $g : \mathbb{C} \dashrightarrow \mathbb{C}$, with set of poles $= B$, such that for each $n \geq 1$, the Laurent series expansion of g, near b_n is

$$g(z) = P_n\left(\frac{1}{z-b_n}\right) + \text{holomorphic/regular part near } b_n.$$

We will prove these theorems next week.

(3) The choice of circles $C(0; R_m)$ in the hypotheses §1 above (or in Thm on page 3) is a matter of convenience. One could relax this to the following:

For $m \in \mathbb{Z}_{\geq 1}$, it is possible to choose a contour $\mathcal{C}_m$ in $\mathbb{C} \setminus A$ in such a manner that

- $\text{Interior}(\mathcal{C}_1) \subset \text{Interior}(\mathcal{C}_2) \subset \dots$ $\quad \mathbb{C} = \bigcup_{m=1}^{\infty} \text{Interior}(\mathcal{C}_m)$
- for a fixed $z_0 \in \mathbb{C}$, $\text{distance}(z_0, \mathcal{C}_m) \to \infty$ as $m \to \infty$.
- $\exists$ ~~a~~ constants $M > 0$, $C > 0$ such that

$$|f(z)| < M \cdot e^{C|z|} \quad \forall z \in \mathcal{C}_m.$$

§3. Jump behaviour of $\{\mathcal{L}_\theta f(z)\}_\theta$ - Also known as

Stokes' phenomenon for Laplace-type integrals.

$f : \mathbb{C} \dashrightarrow \mathbb{C}$, $0 \notin A \subset \mathbb{C}$ poles of f, satisfying hypotheses §1 above.

(Exercise - $f(t) = \dfrac{t}{e^t - 1}$ satisfies these hypotheses).

Theorem. - For each $\theta \in \mathbb{R}$ such that $\ell_{-\theta} \cap A = \phi$,

$\mathcal{L}_\theta f(z)$ is a holomorphic function on $\mathbb{H}_{\theta, c}$.

If $\theta_1 < \theta_2 \leq \theta_1 + \pi$, are two real numbers such that

$\ell_{-\theta_1} \cap A = \ell_{-\theta_2} \cap A = \phi$,

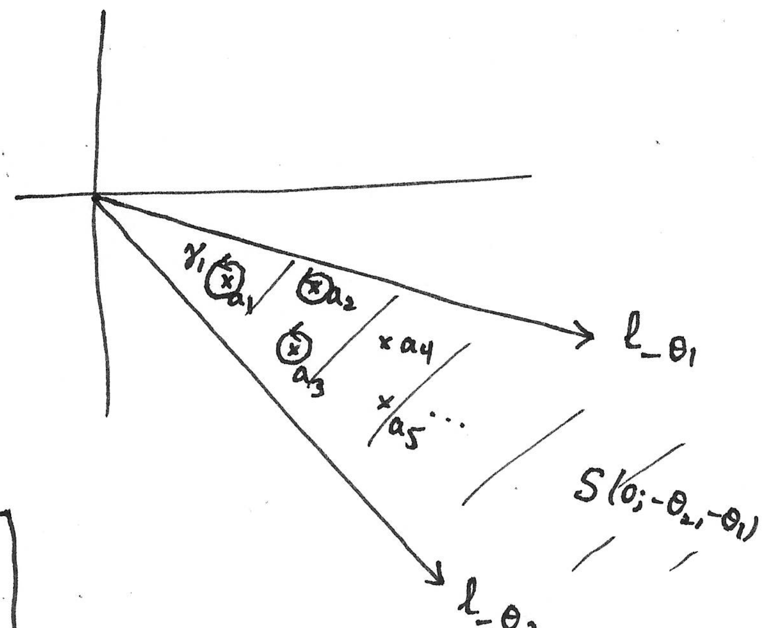
and points in $A \cap S(0; -\theta_2, -\theta_1)$

$= \{a_1, a_2, \dots\}$

are arranged in ascending moduli

$(0 < |a_1| \leq |a_2| \leq \dots)$,

then $\forall z \in \mathbb{H}_{\theta_1, c} \cap \mathbb{H}_{\theta_2, c}$



$$\boxed{\mathcal{L}_{\theta_2} f(z) - \mathcal{L}_{\theta_1} f(z) = \sum_{k=1}^{\infty} \int_{\gamma_k} f(t) e^{-zt}\, dt}$$

($\gamma_1, \gamma_2, \dots$ are small circles around $a_1, a_2, \dots$

s.t. $\text{Interior}(\gamma_k) \cap A = \{a_k\}$

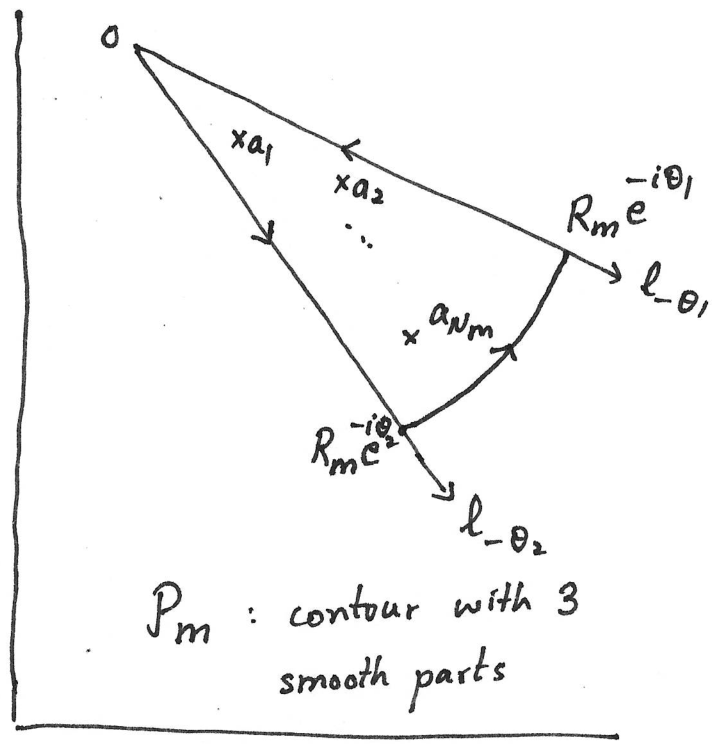
$\text{Interior}(\gamma_k) \subset S(0; -\theta_2, -\theta_1)$)

Proof. Pick $C_1 > C$ and assume $z \in \mathbb{H}_{\theta_1, C} \cap \mathbb{H}_{\theta_2, C}$ is such that $\mathrm{Re}(z e^{-i\phi}) \geq C_1 \quad \forall \phi \in (\theta_1, \theta_2)$

(cannot be done if $\theta_2 - \theta_1 \geq \pi$)

For each $m \in \mathbb{Z}_{\geq 1}$ let $\mathcal{P}_m$ be the following contour

- $0 \to R_m e^{-i\theta_2}$ straight line segment
- $\mu_m : R_m e^{-i\theta_2} \to R_m e^{-i\theta_1}$ along circular arc
- $R_m e^{-i\theta_1} \to 0$ line segment



$\mathcal{P}_m$: contour with 3 smooth parts

By our hypotheses §1, f has no poles on $\mathcal{P}_m$ and

$$|f(t)| < M e^{C R_m} \quad \text{for } t \in \mu_m.$$

Let $\{a_1, a_2, \ldots, a_{N_m}\} = A \cap \text{Interior}(\mathcal{P}_m)$.

By principle of contour deformation,

$$\int_{\mathcal{P}_m} f(t) e^{-zt} \, dt = \sum_{k=1}^{N_m} \int_{\gamma_k} f(t) e^{-zt} \, dt \qquad -(1)$$

L.H.S. of (1) $= \int_0^{R_m e^{i\theta_2}} + \int_{\mu_m}^{R_m e^{i\theta_2}} - \int_0^{R_m e^{i\theta_1}} f(t) e^{-zt}\, dt$

Now $\left| \int_{\mu_m} f(t) e^{-zt}\, dt \right| < M e^{-(c_1 - c) R_m} \to 0$ as $m \to \infty$

(since $R_m \to \infty$ as $m \to \infty$).

So, $\lim_{m \to \infty}$ of (1) gives

$$\mathcal{L}_{\theta_2} f(z) - \mathcal{L}_{\theta_1} f(z) = \sum_{k=1}^{\infty} \int_{\gamma_k} f(t) e^{-zt}\, dt$$

as claimed. $\square$