§1. Generalized partial fractions. - (Mittag-Leffler 1880).

Let $f : \mathbb{C} \dashrightarrow \mathbb{C}$ be a meromorphic function; $A \subset \mathbb{C}$ its set of poles. We assume that $0 \notin A$, and every $a \in A$ is a simple pole (i.e., pole of order 1) and let $\rho_a = \operatorname{Res}_{z=a} (f)$.

<u>Main hypothesis</u> : It is possible to choose contours $\mathcal{C}_1, \mathcal{C}_2, \dots$ in $\mathbb{C} \setminus A$, so as to satisfy:

(i) $\text{Interior}(\mathcal{C}_1) \subset \text{Interior}(\mathcal{C}_2) \subset \dots$; $\mathbb{C} = \bigcup_{m=1}^{\infty} \text{Interior}(\mathcal{C}_m)$

(ii) $\forall z \in \mathbb{C}$, $\lim_{m\to\infty} \text{distance}(z; \mathcal{C}_m) = \infty$. $\lim_{m\to\infty} \dfrac{\text{length}(\mathcal{C}_m)}{\text{dist}(z; \mathcal{C}_m)} < \infty$.

(iii) $\exists M > 0$ s.t. $|f(z)| \le M \quad \forall z \in \bigcup_{m=1}^{\infty} \mathcal{C}_m$.

Then :

$$f(z) = f(0) + \sum_{a \in A} \rho_a \left(\frac{1}{z-a} + \frac{1}{a} \right)$$

(here the possibly infinite sum $\sum_{a\in A}$ is to be arranged in the ascending order of moduli: $A = \{a_1, a_2, a_3, \dots\}$,
$0 < |a_1| \le |a_2| \le \dots$

and $\displaystyle\sum_{a \in A} = \lim_{N\to\infty} \sum_{k=1}^{N}$ (over $a_1, \dots, a_N$) .)

Proof.– Let $\Omega = \mathbb{C} \setminus A$, so $f : \Omega \to \mathbb{C}$ is holomorphic.

For each $m \geq 1$, let $N_m \in \mathbb{Z}_{\geq 1}$ denote the positive integer so that

$$a_1, a_2, \ldots, a_{N_m} \in \text{Interior}(\mathcal{C}_m)$$

$$a_n \in \text{Exterior}(\mathcal{C}_m) \quad \forall\, n > N_m.$$

Let $w \in \Omega$. By Cauchy's integral formula ($m \gg 0$ s.t. $w \in$ Interior of $\mathcal{C}_m$).

$$\frac{1}{2\pi i} \int_{\mathcal{C}_m} \frac{f(z)}{z-w}\, dz = f(w) + \sum_{k=1}^{N_m} \frac{\rho_k}{a_k - w} \qquad -(1)$$

($\rho_k = \rho_{a_k}$ = residue of f at a_k)

Replace $\displaystyle\frac{1}{z-w} = \frac{1}{z}\left(1 + \frac{w}{z-w}\right)$ to get:

$$\frac{1}{2\pi i} \int_{\mathcal{C}_m} \frac{f(z)}{z-w}\, dz = \frac{1}{2\pi i} \int_{\mathcal{C}_m} \frac{f(z)}{z}\, dz + \frac{w}{2\pi i} \int_{\mathcal{C}_m} \frac{f(z)}{z(z-w)}\, dz$$

$$= f(0) + \sum_{k=1}^{N_m} \frac{\rho_k}{a_k} + \frac{w}{2\pi i} \int_{\mathcal{C}_m} \frac{f(z)}{z(z-w)}\, dz \qquad -(2)$$

Combining (1) and (2), we get

$$f(w) = f(0) + \sum_{k=1}^{N_m} \rho_k \left(\frac{1}{w - a_k} + \frac{1}{a_k} \right) + \frac{w}{2\pi i} \int_{\mathcal{C}_m} \frac{f(z)}{z(z-w)}\, dz$$

Claim : $$\lim_{m\to\infty} \frac{w}{2\pi i}\int_{\mathcal{C}_m} \frac{f(z)}{z(z-w)}\,dz = 0$$ uniformly in $w\in\Omega$ (cpct subsets of)

Let K be a compact set, $K\subset\Omega$, and $\varepsilon>0$ be given.

We need to find $N\gg 0$ s.t. $\left|\frac{w}{2\pi i}\int_{\mathcal{C}_m}\frac{f(z)}{z(z-w)}\,dz\right| < \varepsilon$, $\forall w\in K$, $m\geq N$.

First, let $A\in\mathbb{R}_{>0}$ be s.t. ~~$|z|<A\ \forall z\in K$~~.
$|w|<A$, $\forall w\in K$.

Let $d_{K,m} = \text{distance}(K,\mathcal{C}_m)$.

Then,

$$\left|\frac{w}{2\pi i}\int_{\mathcal{C}_m}\frac{f(z)}{z(z-w)}\,dz\right| \leq \frac{A}{2\pi}\;\frac{M\cdot\text{length}(\mathcal{C}_m)}{d_{K,m}\;\text{distance}(0,\mathcal{C}_m)}$$

As $\lim_{m\to\infty}\frac{\text{length}\,\mathcal{C}_m}{\text{distance}(0,\mathcal{C}_m)}$ is finite, there is $C>0$ s.t.
$\text{length}(\mathcal{C}_m) < C\cdot\text{distance}(0,\mathcal{C}_m)$ $(\forall m)$.

As $\lim_{m\to\infty} d_{K,m} = \infty$, pick $N\gg 0$

s.t. $\frac{A\cdot M\cdot C}{d_{K,m}} < \varepsilon$ for each $m\geq N$. $\square$

§2. Example.

$$\operatorname{cosec}(z) = \frac{1}{z} + \sum_{n \in \mathbb{Z}_{\neq 0}} (-1)^n \left(\frac{1}{z - n\pi} + \frac{1}{n\pi} \right)$$

$$f(z) = \operatorname{cosec}(z) - \frac{1}{z}$$ (so zero is not a pole)

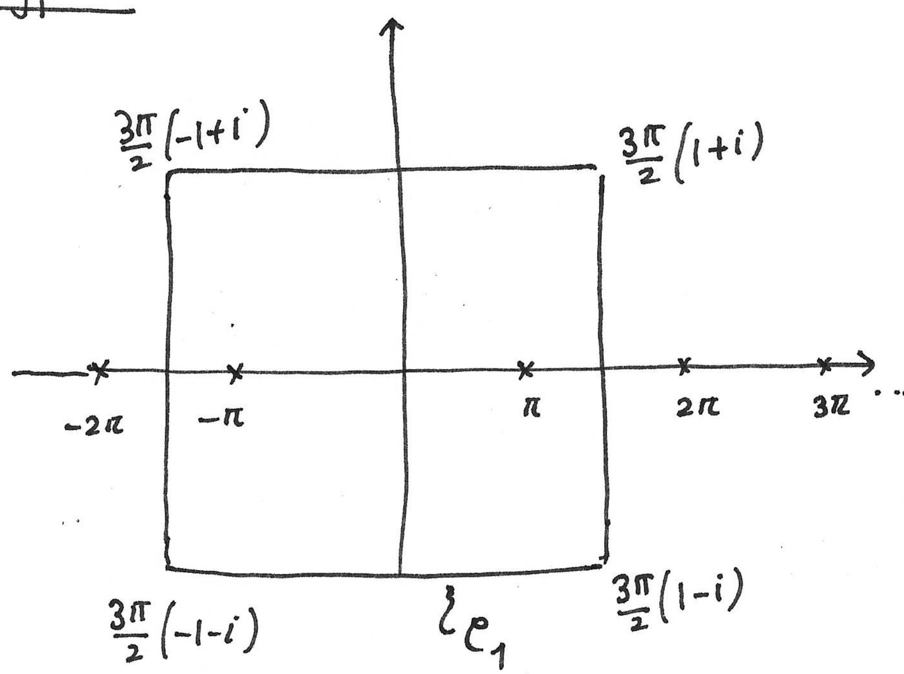
- $f(0) = \lim_{x \to 0} \left(\frac{1}{\sin(x)} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x - \sin(x)}{x \sin(x)} = 0$

- f has simple poles at $\pi \cdot \mathbb{Z}_{\neq 0}$.

$$\operatorname{Res}_{n\pi}(f) = \operatorname{Res}_{n\pi}\left(\frac{1}{\sin(z)} \right) = \lim_{x \to n\pi} \frac{x - n\pi}{\sin(x)} = (-1)^n.$$

Verification of the main hypothesis:

$\mathcal{C}_m$ = square with vertices $\left(\frac{2m+1}{2}\right)(\pm 1 \pm i)\pi$



$$\left| \operatorname{cosec}(z) \right| = \frac{2}{|e^{iz} - e^{-iz}|} \leq \frac{2}{\left| |e^{iz}| - |e^{-iz}| \right|} = \frac{2}{|e^{y} - e^{-y}|}$$

$y = \operatorname{Im}(z).$

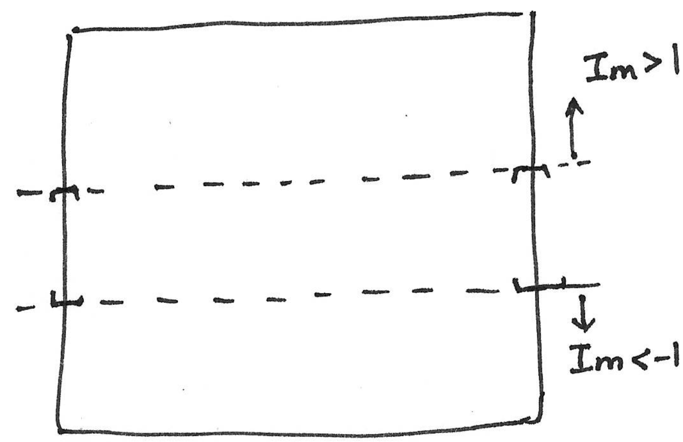
So, for instance, if $|\mathrm{Im}(z)| > 1$, then

$$|\operatorname{cosec}(z)| < \frac{2}{e-1}.$$

$\mathcal{C}_m \cap \{|\mathrm{Im}| \leq 1\}$

consists of 2 vertical segments

$$i\cdot[-1,1] \pm \frac{2m+1}{2}\pi.$$



$\mathcal{C}_m$

Note:
~~As~~ $|\operatorname{cosec}(z)|$ is periodic in $\mathbb{Z}\pi$

Now we can pick a constant $M>0$ s.t. $|\operatorname{cosec}(z)| < M$

$$\forall z \in \left\{\frac{\pi}{2}+it : -1 \leq t \leq 1\right\}$$

By periodicity $|\operatorname{cosec}(z)| < M$

$$\forall z \in \left\{\pm(2m+1)\frac{\pi}{2} + it : -1 \leq t \leq 1\right\}.$$

So $|\operatorname{cosec}(z)|$ is bounded by $\mathrm{Max}\left\{M, \frac{2}{e-1}\right\}$ ~~as z~~

$$\forall z \in \bigcup_{m=1}^{\infty} \mathcal{C}_m. \qquad \square$$

§3. Weierstrass' infinite product theorem

Assume g is an entire non-constant holomorphic function, $g: \mathbb{C} \to \mathbb{C}$

Let $A = \{a \in \mathbb{C} : g(a) = 0\} \subset \mathbb{C}$ discrete set.

Assume that $0 \notin A$, and every $a \in A$ is a simple zero (i.e. $g'(a) \neq 0 \ \forall a \in A$).

Assume our main hypothesis (see page 1 above) holds for $f = \frac{g'}{g}$. Then,

$$g(z) = g(0)\, e^{\frac{g'(0)}{g(0)} z} \prod_{a \in A} \left(1 - \frac{z}{a}\right) e^{z/a}$$

<u>Proof.</u>- By Theorem §1 above, applied to $g'/g = f$, we get

$$f(z) = f(0) + \sum_{a \in A} \rho_a \left(\frac{1}{z-a} + \frac{1}{a} \right)$$

$$\rho_a = \operatorname{Res}_a \left(\frac{g'}{g} \right) = \text{order of vanishing of } g \text{ at } a = 1.$$

Hence $g : \mathbb{C} \to \mathbb{C}$ solves the following differential equation

$$\frac{g'(z)}{g(z)} = f(0) + \sum_{a \in A} \left(\frac{1}{z-a} + \frac{1}{a} \right)$$

R.H.S. is a uniformly convergent sum (see Claim on page 3 above) so termwise integration is permitted.

$$g(z) = C \cdot e^{f(0)\,z} \cdot \prod_{a \in A} \left(1 - \frac{z}{a}\right) e^{z/a} .$$

□

$C = g(0)$.

Exercise. – Do Example §2 for $\cot(z)$, to get

$$\cot(z) = \frac{1}{z} + \sum_{n \in \mathbb{Z}_{\neq 0}} \left(\frac{1}{z - n\pi} + \frac{1}{n\pi} \right)$$

and finally,

$$\frac{\sin(z)}{z} = \prod_{n \in \mathbb{Z}_{\neq 0}} \left(1 - \frac{z}{n\pi}\right) e^{z/n\pi}$$

$$= \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2}\right) .$$