§1. Mittag-Leffler Theorem.- Assume that we are given

(1) A discrete set $A \subset \mathbb{C}$. $0 \notin A$.

$$A = \{a_1, a_2, a_3, \dots\} \qquad 0 < |a_1| \leq |a_2| \leq \dots$$

(2) $\forall\, n \geq 1$, a polynomial $P_n(z) \in \mathbb{C}[z]$.

Then, $\exists\, f : \mathbb{C} \dashrightarrow \mathbb{C}$ meromorphic, with set of poles of $f = A$; and

$\forall\, n \geq 1$, Laurent Series expansion of f near a_n

$$= P_n\left(\frac{1}{z - a_n}\right) + \text{holomorphic part.}$$

Note: This f need not be unique. Clearly $f + g$ ($g : \mathbb{C} \to \mathbb{C}$ entire hol. fn.) again fulfills the criteria of the theorem.

Examples.- (1) Let $A = \{n^2 : n \in \mathbb{Z}_{\geq 1}\}$; $P_n(z) = z$.

Singular part near n^2 is $\dfrac{1}{z - n^2}$

Easy exercise $\displaystyle\sum_{n=1}^{\infty} \frac{1}{z - n^2}$ converges uniformly on cpct subsets in $\mathbb{C} \setminus A$.

(Hint $\displaystyle\frac{1}{z - n^2} = -\frac{1}{n^2}\left(1 - n^{-2} z\right)^{-1} = -\sum_{k=0}^{\infty} n^{-2(k+1)} z^k$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent)

(2) Let $A = \mathbb{Z}_{\geq 1}$ and $P_n(z) = \alpha_n^{-1} z$ $\quad (\alpha_1, \alpha_2, \dots \in \mathbb{C}^{\times})$.

Singular part near $z = n$ $= \dfrac{\alpha_n}{z - n}$.

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{z-n}$$ is not, in general, convergent.

e.g. when $\alpha_n = 1 \ \forall n \geq 1$. But $$\sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{n}\right)$$ is.

(3) $A = \{\sqrt{n} : n \in \mathbb{Z}_{\geq 1}\}$; $P_n(z) = z \quad \forall n \geq 1$.

$$\sum_{n=1}^{\infty} \frac{1}{z-\sqrt{n}}$$ ~~is~~ not convergent.

$$\sum_{n=1}^{\infty}\left(\frac{1}{z-\sqrt{n}} + \frac{1}{\sqrt{n}}\right)$$ still not convergent.

$$\sum_{n=1}^{\infty}\left(\frac{1}{z-\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{z}{n}\right)$$ convergent ✓.

§2. <u>Proof of Mittag-Leffler Theorem.</u>

Consider the Taylor series expansion of $P\left(\frac{1}{z-a}\right)$ near $z=0$ $(a \in A;\ 0 \notin A)$.

$$P_n\left(\frac{1}{z-a_n}\right) = \sum_{l=0}^{\infty} c_l^{(n)} z^l .$$ radius of convergence $= |a_n|$.

For each $n \geq 1$, pick $M_n > 0$ s.t.

$$\left| P_n\left(\frac{1}{z-a_n}\right) - \sum_{l=0}^{M_n} c_l^{(n)} z^l \right| < \frac{1}{2^n}, \quad \forall z \text{ s.t. } |z| \leq \frac{|a_n|}{2}.$$

Define $$f(z) = \sum_{n=1}^{\infty}\left(P_n\left(\frac{1}{z-a_n}\right) - \sum_{l=0}^{M_n} c_l^{(n)} z^l\right)$$

<u>Claim.</u> – $\displaystyle\sum_{n=1}^{\infty}\Bigg(P_n\left(\frac{1}{z-a_n}\right)-\underbrace{\sum_{\ell=0}^{M_n} c_\ell^{(n)} z^\ell}_{C_n(z)}\Bigg)$ converges uniformly rel. to cpct subsets of $\mathbb{C}\setminus A$.

<u>Proof.</u> – Let $K\subset \mathbb{C}\setminus A$, K compact, and $\varepsilon>0$ be given. Let $R>0$ be s.t. $|z|<R\ \forall z\in K$.

Choose $N\gg 0$ so that
$$2^N > \frac{1}{\varepsilon}$$
$$|a_n| > 2\cdot R \quad \forall n\geq N.$$

Then, $\forall\ n\geq N$ and $m\geq 0$, $z\in K$ we have:

$$\left|\sum_{k=n}^{n+m}\left(P_k\left(\frac{1}{z-a_k}\right)-C_k(z)\right)\right| \leq \sum_{k=n}^{n+m}\frac{1}{2^k} < \sum_{k=N+1}^{\infty}\frac{1}{2^k}$$
$$= \frac{1}{2^N} < \varepsilon.$$

□

§3. <u>Euler's gamma function</u> (Euler, 1729).

The problem of interpolating points $(n, n!)$ $(n\in\mathbb{N})$ was posed by Goldbach in 1720's. Euler's solution was based on the following calculation
$$\int_0^\infty t^n e^{-t}\,dt = n! \qquad \forall n\geq 0.$$

Definition (the notation, and -1 shift in the exponent of t, are due to Lagrange)
$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt.$$

Theorem. - $\int_0^\infty t^{z-1} e^{-t}\, dt$ converges uniformly on (cpct subsets of) $\{Re > 0\}$ half plane

Hence, $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$ is a hol. fn. $\Gamma : \mathbb{H} \to \mathbb{C}$
($\mathbb{H}$ = right half plane).

Proof. We need to prove (see Lecture 13, Thm §1):

Given a cpct set $K \subset \mathbb{H}$ and $\varepsilon > 0$, $\exists\ r, R$ s.t.

$$\left| \int_0^s t^{z-1} e^{-t}\, dt \right| < \varepsilon, \quad \forall z \in K,\ s < r. \qquad -(1)$$

$$\left| \int_S^\infty t^{z-1} e^{-t}\, dt \right| < \varepsilon, \quad \forall z \in K,\ S > R. \qquad -(2)$$

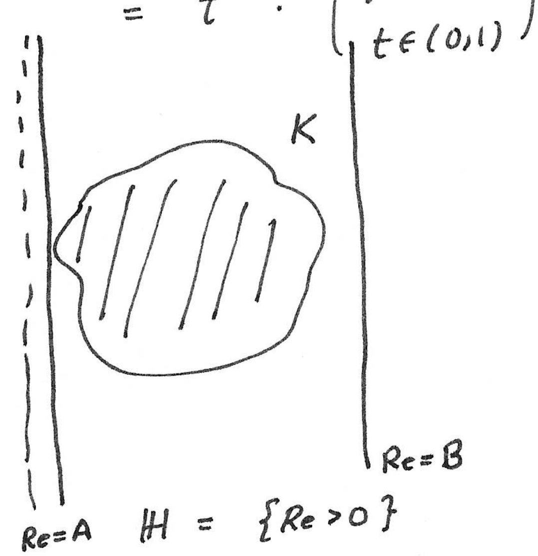
(1) $\quad |t^{z-1}| = |e^{(z-1)\ln(t)}| = e^{Re((z-1)\ln(t))}$

For $t \in (0,1)$, $\ln(t) < 0$. If $A < Re(z)$ $\forall z \in K$,

then $|t^{z-1}| = e^{Re(z-1)\cdot \ln(t)} < e^{(A-1)\ln(t)} = t^{A-1}$. $\left(\begin{matrix}\forall z \in K \\ t \in (0,1)\end{matrix}\right)$

Pick $r \in (0,1)$ s.t. $r^A < \varepsilon A$.

Then $\left| \int_0^s t^{z-1} e^{-t} dt \right| \leq \int_0^r t^{A-1} dt = \dfrac{r^A}{A} < \varepsilon.$



(2) $$\left| t^{z-1} \right| = e^{\ln(t)\cdot \mathrm{Re}(z-1)} < t^{B-1} \quad \text{for } t>1.$$

(here $B>0$ is so that $\mathrm{Re}(z) < B \;\; \forall z \in K$)

As $t^{B-1} e^{-t/2} \to 0$ as $t \to \infty$,

$\exists\, t_0 \in \mathbb{R}_{>0}$ s.t. $t^{B-1} e^{-t/2} \leqslant 1 \;\; \forall t > t_0$.

Pick $R > t_0$ such that $e^{-R/2} < \frac{\varepsilon}{2}$. Then, $\forall_{z \in K} S > R$ we have

$$\left| \int_S^\infty t^{z-1} e^{-t}\, dt \right| \leqslant \int_S^\infty \big(\underbrace{t^{B-1} e^{-t/2}}_{\leqslant 1}\big) e^{-t/2}\, dt$$

$$\leqslant \int_R^\infty e^{-t/2}\, dt = 2 \cdot e^{-R/2} < \varepsilon.$$

$\square$

§4. $\boxed{\Gamma(z+1) = z\,\Gamma(z)}$ $\forall z \in \mathbb{H}$. $(\mathbb{H} = \{\mathrm{Re} > 0\})$.

Proof. (integration by parts)

$$\Gamma(z+1) - z\,\Gamma(z) = \int_0^\infty \left(t^z - z \cdot t^{z-1}\right) e^{-t}\, dt$$

$$= \int_0^\infty \frac{d}{dt}\left(-t^z e^{-t}\right) dt = \lim_{t\to 0^+} t^z e^{-t} - \lim_{t \to \infty} t^z e^{-t}$$

$$= 0.$$

$\square$

Using this difference equation, we can extend Γ to a mero. fn. $\Gamma : \mathbb{C} \dashrightarrow \mathbb{C}$ as follows.

For each $N \in \mathbb{Z}_{\geq 0}$, let $\mathbb{H}_N = \{ z \in \mathbb{C} : \begin{array}{l} z+N \in \mathbb{H} \\ \text{i.e. } Re(z) > -N \end{array} \}$

$$\Gamma(z+N) = (z+N-1)\Gamma(z+N-1) = \cdots = (z+N-1)\cdots z\,\Gamma(z).$$

So. $\dfrac{\Gamma(z+N)}{z(z+1)\cdots(z+N-1)} : \mathbb{H}_N \setminus \{0,-1,\dots,-N+1\} \to \mathbb{C}$ is hol. w/ simple poles at $\{0,-1,\dots,-N+1\}$.

$\rightsquigarrow$ $\Gamma : \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \longrightarrow \mathbb{C}$ hol. fn. with simple poles at $\mathbb{Z}_{\leq 0}$.

§5. $\Gamma(1) = 1$ and $\Gamma(z+1) = z\Gamma(z) \Rightarrow \Gamma(n) = (n-1)!$ $\forall\, n \in \mathbb{Z}_{\geq 1}$.

Residue calculation.

$$\operatorname*{Res}_{z=0} \Gamma(z) = \lim_{z\to 0} z\Gamma(z) = \Gamma(z+1)\Big|_{z=0} = \Gamma(1) = 1.$$

$$\operatorname*{Res}_{z=-n} \Gamma(z) = \lim_{z\to -n} (z+n)\Gamma(z) = \lim_{w\to 0} w\Gamma(w-n)$$

Now $\Gamma(w-n) = \dfrac{\Gamma(w)}{(w-n)\cdots(w-1)}$; so $\operatorname*{Res}_{z=-n}(\Gamma(z)) = \lim_{w\to 0} \dfrac{\Gamma(w+1)}{(w-n)\cdots(w-1)}$

$$= \frac{(-1)^n}{n}.$$

§6. Euler's product formula.

$$\Gamma(z) = \lim_{N\to\infty} \frac{N!}{z(z+1)\cdots(z+N)} \cdot N^z$$

$$= \lim_{N\to\infty} \int_0^N t^{z-1}\left(1-\frac{t}{N}\right)^N dt$$

Proof.- Let $G_N(z) = \int_0^N t^{z-1}\left(1-\frac{t}{N}\right)^N dt$.

We will now prove that $\lim_{N\to\infty} G_N(z) = \Gamma(z)$ uniformly on cpct subsets of $\mathbb{H}$.

$$\Gamma(z) - G_N(z) = \int_0^\infty t^{z-1} e^{-t}\,dt - \int_0^N t^{z-1}\left(1-\frac{t}{N}\right)^N dt$$

$$= \int_N^\infty t^{z-1}e^{-t}\,dt + \int_0^N t^{z-1}\left(e^{-t} - \left(1-\frac{t}{N}\right)^N\right) dt$$

Inequality: $\quad 0 \leqslant e^{-t} - \left(1-\frac{t}{N}\right)^N \leqslant e^{-t}\frac{t^2}{N} \quad \forall\, t\in[0,N]$

(assume for now, proof in the next section)

So $$\left|\Gamma(z) - G_N(z)\right| \leqslant \left|\int_N^\infty t^{z-1}e^{-t}\,dt\right| + \int_0^\infty t^{\mathrm{Re}(z)-1}\frac{t^2}{N}e^{-t}\,dt$$

$$= \left| \int_N^\infty t^{z-1} e^{-t}\,dt \right| + \frac{\Gamma(\mathrm{Re}(z)+2)}{N}.$$

For a cpct set K, choose $N \gg 0$ s.t. $\left| \int_N^\infty t^{z-1} e^{-t}\,dt \right| < \frac{\varepsilon}{2}$, $\forall z \in K$.

and $\frac{M}{N} < \frac{\varepsilon}{2}$, where $M = \mathrm{Max}\{|\Gamma(\mathrm{Re}(z)+2)| : z \in K\}$

Then $|\Gamma(z) - G_{n}(z)| < \varepsilon$, $\forall\, n \geq N$ and $z \in K$.

$\square$ (of $\lim_{N\to\infty} G_N = \Gamma$)

Now we will prove that

$$G_N(z) = \frac{N!}{z(z+1)\cdots(z+N)} \cdot N^z.$$

$$G_N(z) = \int_0^N t^{z-1}\left(1-\frac{t}{N}\right)^N dt \underset{(t=N\tau)}{=} N^{z-1}\cdot N \int_0^1 \tau^{z-1}(1-\tau)^N d\tau$$

So $$H_N(z) = N^{-z} G_N(z) = \int_0^1 \tau^{z-1}(1-\tau)^N d\tau.$$

$$H_0(z) = \int_0^1 \tau^{z-1}\,d\tau = \left.\frac{\tau^z}{z}\right|_{\tau=0}^{1} = \frac{1}{z}.$$

$$H_{n+1}(z) = \int_0^1 \tau^{z-1}(1-\tau)^{n+1} d\tau = \left[(1-\tau)^{n+1}\frac{\tau^z}{z}\right]_0^1 + \int_0^1 \frac{\tau^z}{z}(n+1)(1-\tau)^n d\tau$$

$$= \frac{n+1}{z} H_n(z+1)$$

$$= \frac{(n+1)\,n}{z(z+1)} H_{n-1}(z+2)$$

$$= \cdots = \frac{(n+1)\cdots 1}{z(z+1)\cdots(z+n)} H_0(z+n+1)$$

$$= \frac{(n+1)!}{z(z+1)\cdots(z+n+1)}. \qquad \square$$

§7. Proof of $0 \leq e^{-t} - \left(1-\frac{t}{n}\right)^n \leq e^{-t}\frac{t^2}{n} \quad \forall\, t \in [0,n]$.

Note
$$e^x = 1 + x + \frac{x^2}{2!} + \cdots \geq 1 + x$$
$$e^x = 1 + x + \frac{x^2}{2!} + \cdots \leq 1 + x + x^2 + \cdots = \frac{1}{1-x} \qquad \forall\, x \in [0,1]$$

Take inverse to get $\quad 1 - x \leq e^{-x} \leq \frac{1}{1+x} \quad \forall\, x \in [0,1]$

set $x = \frac{t}{n}$ and take n-th power: $\quad \left(1-\frac{t}{n}\right)^n \leq e^{-t} \leq \left(1+\frac{t}{n}\right)^{-n}$

$$\Rightarrow 0 \leq e^{-t} - \left(1-\frac{t}{n}\right)^n = e^{-t}\left(1 - e^{t}\left(1-\frac{t}{n}\right)^n\right)$$
$$\leq e^{-t}\left(1 - \left(1+\frac{t}{n}\right)^n\left(1-\frac{t}{n}\right)^n\right)$$
$$= e^{-t}\left(1 - \left(1-\frac{t^2}{n^2}\right)^n\right) \leq e^{-t}\frac{t^2}{n} \qquad \square$$

used: $(1-y)^n \geq 1 - ny \quad \forall\, n \geq 0,\ y \in [0,1]$