Lecture 18

Gamma function cntd.

Recall - last time we introduced $\Gamma : \mathbb{C} \dashrightarrow \mathbb{C}$ a meromorphic function, with simple poles at $\mathbb{Z}_{\leq 0}$ and

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\, dt \qquad \forall\ z \text{ so that } Re(z) > 0.$$

Difference equation: $\boxed{\Gamma(z+1) = z\Gamma(z)}$

Residues: $\underset{z=-n}{\operatorname{Res}} \Gamma(z) = \dfrac{(-1)^n}{n!}$; $\Gamma(n) = (n-1)!$

$(\forall\ n \in \mathbb{Z}_{\geq 0})$ $\qquad (n \geq 1)$

Euler's product formula

$$\Gamma(z) = \lim_{N\to\infty} \int_0^N t^{z-1}\left(1 - \frac{t}{N}\right)^N dt = \lim_{N\to\infty} N^z \frac{N!}{z(z+1)\cdots(z+N)}$$

(uniformly in $z \in \underset{\substack{\uparrow \\ \text{cpct.}}}{K} \subset \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$)

§1. Euler-Mascheroni constant $\gamma \approx 0.5772\ldots$

$$\Gamma'(1) = -\lim_{N\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} - \ln(N)\right) = -\gamma$$

(definition of γ).

Proof. Let $G_N(z) = N^z \dfrac{N!}{z(z+1)\cdots(z+N)}$

Take logarithmic-derivative to get

$$\frac{G_N'(z)}{G_N(z)} = \ln(N) - \sum_{K=0}^{N} \frac{1}{z+K}$$

$\lim_{N\to\infty}$ of this equation, and the fact that $\Gamma(1)=1$ gives

$$\Gamma'(1) = -\gamma. \qquad \square$$

Ex: Prove directly that $\left\{ \sum_{K=1}^{N} \frac{1}{K} - \ln(N) \right\}_{N=1}^{\infty}$ converges.

(Hint: $\frac{1}{n} - \ln(n+1) + \ln(n) = \int_0^1 \frac{t}{n(t+n)}\,dt < \frac{1}{n^2}$.)

§2 Weierstrass' product formula.

$$\Gamma(z) = \frac{1}{z} \cdot e^{-\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n}$$

Proof. $\Gamma(z) = \lim_{N\to\infty} \frac{N!}{z(z+1)\cdots(z+N)} e^{\ln(N)\cdot z}$

$$= \lim_{N\to\infty} \frac{1}{z}\left(\prod_{K=1}^{N}\left(1+\frac{z}{K}\right)\right)^{-1} e^{z(1+\frac{1}{2}+\dots+\frac{1}{N})} e^{z(\ln(N)-1-\frac{1}{2}\cdots-\frac{1}{N})}$$

$$= \lim_{N\to\infty} \frac{1}{z} e^{z(\ln(N)-1-\frac{1}{2}-\dots-\frac{1}{N})} \prod_{K=1}^{N}\left(1+\frac{z}{K}\right)^{-1} e^{z/K}$$

Claim: $\left\{ \prod_{K=1}^{N} \left(1+\frac{z}{K}\right) e^{-z/K} \right\}_{N=1}^{\infty}$ converges uniformly.

Hence, Weierstrass' infinite product formula follows.

<u>Proof of the claim.</u> – Let $K \subset \mathbb{C}$ be a compact set.

Let $N_0 > 0$ be such that $|z| < \frac{N_0}{2}$ $\forall z \in K$.

For $n \geq N_0$, we have

$$\log\left(1+\frac{z}{n}\right) - \frac{z}{n} = \frac{-z^2}{2n^2} + \frac{z^3}{3n^3} - \cdots, \text{ so}$$

$$\left|\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right| \leq \frac{|z|^2}{2n^2}\left(\left|1 - \frac{2z}{3n} + \frac{2z^2}{4n^2} - \cdots\right|\right)$$

$$\leq \frac{N_0^2}{8n^2}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) = \frac{N_0^2}{4n^2}.$$

Thus, given $\varepsilon > 0$, pick $N_1 > N_0$ s.t. $\sum_{N_1}^{\infty} \frac{1}{n^2} < \frac{4\varepsilon}{N_0^2}$

Then $\left|\sum_{k=n}^{n+l} \log\left(1+\frac{z}{k}\right) - \frac{z}{k}\right| < \varepsilon$ $\forall z \in K,\ n \geq N_1$.

□

§3. <u>$\Gamma(z)$ and trigonometric functions.</u> –

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} = \pi \operatorname{cosec}(\pi z).$$

Proof.

$$\frac{\sin(\pi z)}{\pi z} = \left\{\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}\right\}\left\{\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n}\right\}$$

$$= \frac{1}{z\cdot\Gamma(z)\cdot e^{\gamma z}} \quad \frac{1}{(-z)\,\Gamma(-z)\,e^{-\gamma z}}$$

$$\frac{\sin(\pi z)}{\pi} = \frac{1}{\Gamma(z)\,\Gamma(1-z)} \qquad \text{using } \Gamma(1-z) = (-z)\,\Gamma(-z).$$

§4. Computation of $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx \qquad (p,q \in \mathbb{R}_{>0})$

(Euler) $$\boxed{B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}}$$

Proof.- $$\Gamma(p)\,\Gamma(q) = \int_0^\infty\int_0^\infty e^{-(t_1+t_2)}\, t_1^{p-1}\, t_2^{q-1}\, dt_1\, dt_2$$

Change of variables. $x = t_1 + t_2$ $(0 < x < \infty)$; $y = \dfrac{t_1}{t_1+t_2}$ $(0 < y < 1)$

So, $t_1 = xy \qquad t_2 = x(1-y). \qquad dt_1\, dt_2 \rightsquigarrow x\, dx\, dy.$ (check this.)

$$\Gamma(p)\,\Gamma(q) = \int_0^1\int_0^\infty e^{-x}\,(xy)^{p-1}\, x^{q-1}(1-y)^{q-1}\, x\, dx\, dy$$

$$= \int_0^\infty x^{p+q-1}\, e^{-x}\, dx \int_0^1 y^{p-1}(1-y)^{q-1}\, dy$$

$$= \Gamma(p+q)\cdot B(p,q). \qquad \square$$

§5. Various proofs of $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(1) Directly from the integral:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{1}{\sqrt{t}} e^{-t} \, dt = \int_0^\infty 2 \cdot e^{-x^2} \, dx \qquad \begin{cases} x = \sqrt{t} \\ x^2 = t \\ 2x\,dx = dt \end{cases}$$

Gaussian integral $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$.

(2) $\displaystyle\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$. Set $z = \frac{1}{2}$.

(3) $\displaystyle\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 x^{p-1}(1-x)^{q-1} \, dx$. Set $p = q = \frac{1}{2}$

$$\Gamma\left(\frac{1}{2}\right)^2 = \int_0^1 x^{-1/2}(1-x)^{-1/2} \, dx \qquad x = \sin^2(\theta), \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$= 2\int_0^{\pi/2} d\theta = \pi. \qquad dx = 2\sin(\theta)\cos(\theta)\,d\theta$$

§6. $\displaystyle\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} : \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \to \mathbb{C}$ inherits the following properties.

(1) $$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n}\right)$$

(2) $$\psi(z+1) - \psi(z) = \frac{1}{z}.$$

(3) $$\psi(z) - \psi(1-z) = \pi \cot(\pi z)$$

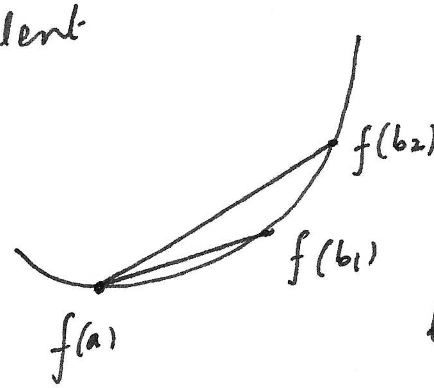
Note: $$\psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2}$$ solves $\psi'(z+1) - \psi'(z) = \frac{-1}{z^2}$.

$$\psi'(x) = \frac{d^2}{dx^2}\left(\log \Gamma(x)\right) > 0 \quad \forall\, x \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}.$$

Hence $\log \Gamma(x)$ is a convex function of $x \in \mathbb{R}_{>0}$

__Defn.__ A continuous function $f : \mathbb{R}_{>0} \to \mathbb{R}$ is __convex__ if $\forall\, a \in \mathbb{R}_{>0}$, the function $b \longmapsto \dfrac{f(b)-f(a)}{b-a}$ is increasing fn. $\forall\, b \in (0,a) \cup (a,\infty)$

Note: if f is twice differentiable, then the definition above is equivalent to $f'' > 0$.



$(b_1 < b_2)$ slopes should be increasing.

§7. Uniqueness of $\Gamma(x)$

Theorem. – Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a continuous function s.t.

(i) $f(x+1) = x\, f(x)$ (ii) $f(1) = 1$

(iii) $\log f$ is a convex function.

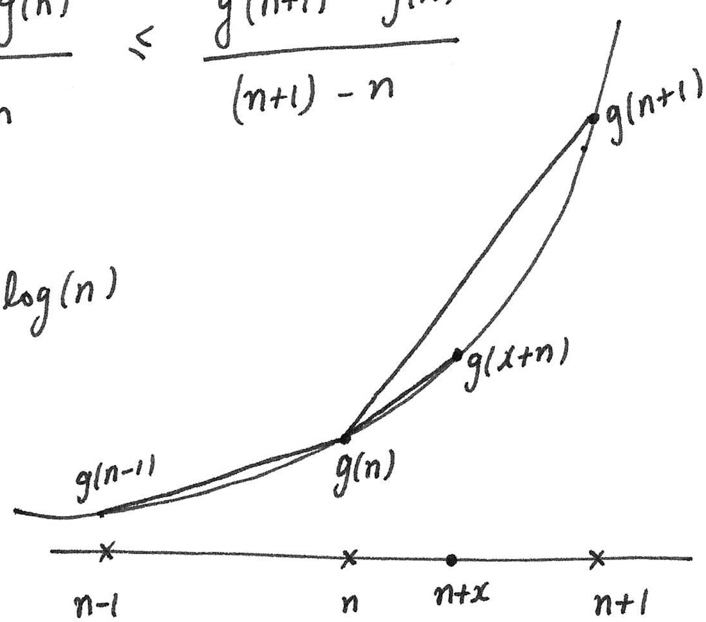
Then $f(x) = \Gamma(x)$ $\forall\, x \in \mathbb{R}_{>0}$.

Proof. – Note, by (i) and (ii) $f(n) = (n-1)!$ $\forall\, n \in \mathbb{Z}_{\geq 1}$.

And $f(x+n) = x(x+1)\cdots(x+n-1)\, f(x)$ – i.e. f is completely determined by its restriction to $(0,1)$.

Let $n \in \mathbb{Z}_{\geq 2}$ and $x \in (0,1)$. By Convexity of $g = \log f$,

$$\frac{g(n-1) - g(n)}{(n-1) - n} \leq \frac{g(x+n) - g(n)}{(x+n) - n} \leq \frac{g(n+1) - g(n)}{(n+1) - n}$$



$$\log(n-1) \leq \frac{g(x+n) - g(n)}{x} \leq \log(n)$$

$$(n-1)^x \leq \frac{f(x+n)}{(n-1)!} \leq n^x$$

$$(n-1)^x \frac{(n-1)!}{x(x+1)\cdots(x+n-1)} \leq f(x) \leq n^x \cdot \frac{(n-1)!}{x(x+1)\cdots(x+n-1)}.$$

Taking $n \to \infty$ limit, we get $f(x) = \Gamma(x)$. $\square$