- If $f: \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is continuous w/
  1) $f(x+1) = x f(x)$
  2) $f(1) = 1$
  3) $\log f$ is convex

Then $f = \Gamma$.

10/9 <u>Reminder</u>

$$\Gamma(z) = \frac{1}{z} e^{-\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$$

$$= \int_0^{\infty} t^{z-1} e^{-t} \, dt$$

$$= \lim_{N \to \infty} N^z \frac{N!}{z(z+1)\dots(z+N)}$$

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln(n)\right) \approx 0.577\dots$$

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{z+n}\right).$$

$$\psi(1) = -\gamma \qquad \zeta(\ell) = \sum_{n=1}^{\infty} \frac{1}{n^{\ell}}$$

$$\psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2} \qquad \psi'(1) = \zeta(2)$$

$$\frac{\psi''(z)}{2} = -\sum_{n=0}^{\infty} \frac{1}{(z+n)^3} \qquad \frac{\psi''(2)}{2!} = -\zeta(3)$$

$$\vdots \qquad \vdots$$

$$\frac{\psi^{(\ell)}(z)}{\ell!} = (-1)^{\ell-1} \sum_{n=0}^{\infty} \frac{1}{(z+n)^{\ell+1}}$$

Taylor Series of $\psi(z)$ near $z=1$

$$\psi(z+1) = \sum_{\ell=0}^{\infty} \frac{\psi^{(\ell)}(z)}{\ell!} z^{\ell}$$

$$= -\gamma + \zeta(2) z - \zeta(3) z^3 + \cdots$$

$$\Rightarrow \log \Gamma(z+1) = \underset{0}{\cancel{C}} - \gamma z + \zeta(2)\frac{z^2}{2} - \zeta(3)\frac{z^3}{3} + \cdots$$

$$\Gamma(z+1) = e^{-\gamma z} \exp\left(\sum_{\ell=1}^{\infty} (-1)^{\ell-1} \frac{\zeta(\ell+1) z^{\ell+1}}{\ell+1} \right)$$

$$\frac{d^2}{dx^2} \log \Gamma(x) > 0 \quad \forall x \in (0,\infty)$$

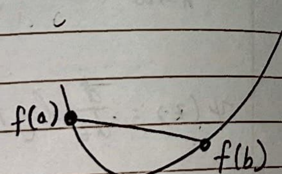
$$\Rightarrow \log \Gamma(x) \text{ is } \underline{\text{Convex}}$$

continuous

- **Defn:** $f: \mathbb{R}_{>0} \to \mathbb{R}$ is continuous convex if

$$\frac{f(b)-f(a)}{b-a}$$

is increasing function of a, b



- **Thm:** If $F: \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is cnts and
  (1) $F(x+1) = x F(x)$
  (2) $F(1) = 1$
  (3) $\log F$ is convex,

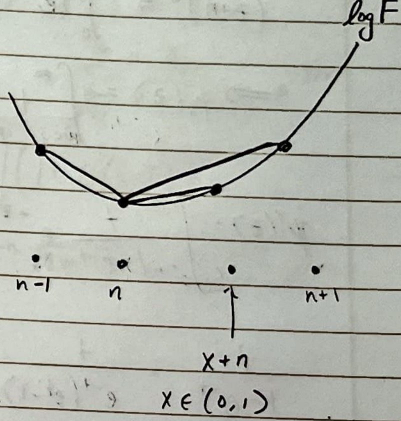
then $F = \Gamma$.

Proof: (1) & (2) $\Rightarrow$ $F(n) = (n-1)!$ $(n \geq 1)$

and

$$F(x+n) = (x+n-1)F(x+n-1)$$
$$= (x+n-1)\ldots xF(x) \quad \forall x \in \mathbb{R}_{>0}.$$

F is uniquely determined by its restriction to $(0,1)$.



$f = \log F$ is convex $\Rightarrow$ $\forall x \in (0,1)$, $n \in \mathbb{N}$

$$\frac{f(n-1)-f(n)}{(n-1)-n} \leq \frac{f(x+n)-f(n)}{x+n-n} \leq \frac{f(n+1)-f(n)}{(n+1)-n}$$

$$\log F(x+n) = \log(x(x+1)\ldots(x+n-1)) + \log F(x)$$

$$\log(n-1)! - \log(n-2) \leq \frac{f(x+n) - \log(n-1)!}{x}$$

$\longrightarrow \Gamma(x)$

$$\frac{x^{n-1}(n-1)!}{x(x+1)\ldots(x+n-1)} \leq F(x).$$

(Complete proof in Friday's notes)

$\square$

Behavior of $\Gamma(x)$ as $x \longrightarrow \infty$

$$\psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}$$

$$\psi'(z+1) - \psi'(z) = -\frac{1}{z^2}$$

$$(z+n)^{-2} = \int_0^{\infty} t e^{-(z+n)t}\, dt$$

$$\Longrightarrow \psi'(z) = \int_0^{\infty} t \sum_{n=0}^{\infty} e^{-(z+n)t}\, dt = \int_0^{\infty} \frac{te^{-zt}}{1-e^{-t}}\, dt$$

$$\psi'(z) = \int_0^{\infty} \frac{t}{1-e^{-t}} e^{-zt}\, dt$$

$$\frac{t}{1-e^{-t}} = \frac{t}{e^{-t}(e^t-1)} = \frac{t(e^t-1)+t}{e^t-1}$$

$$= \frac{t}{e^t-1} + t = 1 + \frac{t}{2} + \sum_{l=1}^{\infty} \frac{B_{2l}}{(2l)!} t^{2l}$$

$$\beta(t) = \frac{t}{e^t-1} = 1 - \frac{t}{2} + \sum_{l=1}^{\infty} B_{2l} \frac{t^{2l}}{(2l)!}$$

$$\psi'(z) \sim z^{-1} + \frac{1}{2} z^{-2} + \sum_{l=1}^{\infty} B_{2l} z^{-2l-1}$$ (Watson's Lemma

divergent series

Integrate term-by-term

$$\Rightarrow \psi(z) \sim \log(z) - \frac{1}{2z} - \sum_{l=1}^{\infty} B_{2l} \frac{z^{-2l}}{2l} + C$$

$$\Rightarrow \log\Gamma(z) \sim z\log(z) - z - \frac{1}{2}\log(z) + \sum_{l=1}^{\infty} B_{2l} \frac{z^{-2l+1}}{2l(2l-1)}$$

$$+ Cz + D.$$

To find constants C, D

$$\gamma(1) = -\gamma \implies C = 0.$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

(Stirling Series)

$$\log \Gamma(z) \sim (z - \tfrac{1}{2}) \log(z) - z + \tfrac{1}{2} \ln(2\pi) + \sum_{\ell=1}^{\infty} \frac{B_{2\ell}}{(2\ell)(2\ell-1)} z^{-2\ell+1}$$

$$\Gamma(z) \sim \boxed{z^{z-\frac{1}{2}} e^{-z} \sqrt{2\pi}} \left(1 + O(z^{-1})\right)$$

$$n! \sim (n+1)^{n+\frac{1}{2}} e^{-n} \sqrt{2\pi}$$

- Method of Steepest Descent

$$\int_a^b g(t) e^{-\frac{f(t)}{\hbar}}$$

$$\sim e^{-f(c)/\hbar} \sqrt{\hbar} \left(\sqrt{2\pi} \frac{g(c)}{\sqrt{f''(c)}} + O(h) \right)$$

OR

$$\int_a^b g(t) e^{-zf(t)} dt \sim \frac{e^{-f(c)z}}{\sqrt{z}} \left(\sqrt{2\pi} \frac{g(c)}{\sqrt{f''(c)}} + O(z^{-1}) \right)$$

as $Re(z) \to \infty$

Hypotheses : $f, g : (a,b) \to \mathbb{R}$ smooth functions
$c \in (a,b)$ where f attains global min
$f''(c) > 0$.

· Application to Γ function

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\, dt \qquad (x \in \mathbb{R}_{\geq 0})$$

(Take $t = x \cdot \tau$)

$$= \int_0^\infty e^{x \overbrace{\ln(x\tau)}^{\ln(x)+\ln(\tau)}} e^{-x\tau}\, x \cdot d\tau$$

$$= x^{x+1} \int_0^\infty e^{-x(\tau - \ln(\tau))}\, d\tau .$$

For our thm, $g(\tau) = 1$, $f(\tau) = \tau - \ln(\tau)$.

To find the min, $f'(\tau) = 1 - \frac{1}{\tau} \Rightarrow c = 1$

$f''(\tau) = \frac{1}{\tau^2} > 0$ ✓

By ThM

$$\Gamma(x+1) \sim x^{x+1} \frac{e^{-x}}{\sqrt{x}} \left(\sqrt{2\pi} + O(x^{-1}) \right)$$

$$x! \sim x^{x+\frac{1}{2}} e^{-x} \sqrt{2\pi} .$$