Lecture 20

§1. Difference operators with constant coefficients.

Let T denote the shift operator $(T\cdot f)(s) = f(s+1)$.

Let $\mathcal{D}(T) \in \mathbb{C}[T]$ be a polynomial in T with non-zero constant term.

Lemma. – Let k = order of vanishing of $\mathcal{D}(T)$ at $T=1$.

Then $\mathcal{D}(T)$ sets up an isomorphism (of vector spaces)

$$s^{-1}\mathbb{C}[[s^{-1}]] \xrightarrow{\sim} s^{-k-1}\mathbb{C}[[s^{-1}]].$$

Proof. Let $F(s) = \sum_{n=0}^{\infty} F_n s^{-n-1}$; and $\mathcal{D}(T) = \sum_{j=0}^{m} d_j T^j$ $(d_0 \neq 0)$.

Then,
$$\mathcal{D}(T)\cdot F(s) = \sum_{n=0}^{\infty} F_n \left(\sum_{j=0}^{m} d_j (s+j)^{-n-1}\right)$$
$$= \sum_{n=0}^{\infty} F_n s^{-n-1}\left(\sum_{j=0}^{m} d_j (1+j\cdot s^{-1})^{-n-1}\right)$$
$$= \sum_{n=0}^{\infty} F_n s^{-n-1}\left(\sum_{j=0}^{m} d_j \left(\sum_{\ell=0}^{\infty} (-1)^\ell \binom{n+\ell}{\ell} j^\ell s^{-\ell}\right)\right)$$
$$= \sum_{N=0}^{\infty} s^{-N-1}\cdot\left(\sum_{\ell=0}^{N} (-1)^\ell \binom{N}{\ell} F_{N-\ell}\cdot\left(\sum_{j=0}^{m} d_j j^\ell\right)\right)$$

Note,
$$\sum_{j=0}^{m} d_j j^\ell = (T\partial_T)^\ell\cdot\mathcal{D}(T)\Big|_{T=1} \quad \begin{array}{l} = 0 \text{ for } 0 \leq \ell \lneq k \\ \neq 0 \text{ for } \ell = k. \end{array}$$

Say $\alpha = (T\partial_T)^k\cdot\mathcal{D}(T)\Big|_{T=1} \neq 0$. by defn. of order of vanishing.

Hence, $\forall\ 0 \le N < k$, the coefficient of s^{-N-1} in $\mathcal{D}(T)\cdot F(s)$ is zero, showing that $\mathcal{D}(T)\cdot F(s) \in s^{-k-1}\cdot\mathbb{C}[\![s^{-1}]\!]$.

Given $G(s) = \sum_{m=k}^{\infty} G_m\, s^{-m-1}$, the equation $\mathcal{D}(T)\cdot F(s) = G(s)$

successively determines coefficients of $F(s)$ uniquely:

$$\forall N \ge 0: \quad \sum_{\ell=k}^{N} (-1)^{\ell} \binom{N}{\ell} F_{N-\ell} \Big(\sum_j d_j\, j^{\ell} \Big) = G_N. \qquad -(*)$$

(holds trivially for $0 \le N < k$: $0 = 0$).

$N = k$. $\quad (-1)^k F_0 \cdot \alpha = G_k \quad \Rightarrow F_0 = (-1)^k \alpha^{-1} G_k$.

In general: $\quad (-1)^k \binom{N}{k} F_{N-k} \cdot \alpha + \underbrace{\text{terms with smaller subscript of } F}_{\text{already determined}} = G_N$

defines F_{N-k} uniquely. Hence, $\forall\, G \in s^{-k-1}\mathbb{C}[\![s^{-1}]\!]$, $\exists!\ F(s) \in s^{-1}\mathbb{C}[\![s^{-1}]\!]$ so that $\mathcal{D}(T)\cdot F(s) = G(s)$. $\square$

§2. Homogeneous case.

Assume $\phi : \mathbb{C} \dashrightarrow \mathbb{C}$ is a mero. fn. so that $\mathcal{D}(T)\cdot\phi = 0$.

Prop. If $\phi(s)$ admits an asymptotic expansion (power series in s^{-1}) as $-\mathrm{Re}(s) \to \infty$ (so, ϕ is assumed to be holomorphic on $\mathrm{Re}(s) \ll 0$), and all roots of $\mathcal{D}(T) = 0$ have modulus ≤ 1, then $\phi \equiv 0$.

(Similarly, if ϕ is hol. on $\mathrm{Re}(s) \gg 0$, has asymptotic power series expansion as $\mathrm{Re}(s) \to +\infty$, and all roots of $\mathcal{D}(T)=0$ have modulus ≥ 1, then $\phi \equiv 0$.)

Proof. – Write $\mathcal{D}(T) = T^m - \sum_{j=0}^{m-1} d_j T^j$ $(d_0 \neq 0)$. Thus, for every $s \in \mathbb{C}$, we have:

$$\phi(s) = \sum_{j=0}^{m-1} d_j \, \phi(s+m+j) .$$

In vector notation,

if $\vec{\Phi}(s) = \begin{bmatrix} \phi(s) \\ \phi(s+1) \\ \vdots \\ \phi(s+m-1) \end{bmatrix}_{m \times 1}$, then $\vec{\Phi}(s) = \underbrace{\begin{bmatrix} d_{m-1} & d_{m-2} & \cdots & d_0 \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 \;\; 1 & 0 \end{bmatrix}}_{\mathbb{D}} \underbrace{\begin{bmatrix} \phi(s+1) \\ \phi(s+2) \\ \vdots \\ \phi(s+m) \end{bmatrix}}_{\vec{\Phi}(s+1)}$

(companion matrix of $\mathcal{D}(T)$)

By assumption on roots of $\mathcal{D}(T)$ (= eigenvalues of $\mathbb{D}$) spectral radius $\rho(\mathbb{D}) \leq 1$. ($\rho(\mathbb{D}) = \mathrm{Max}\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathbb{D}\}$).

Moreover, by Lemma §1, if $\phi(s)$ admits an asymptotic power series expansion, then $\phi(s) \sim 0$ as $\mathrm{Re}(s) \to -\infty$.

Meaning, $\forall\, \ell \geq 1$, we can find $R \gg 0$ such that

$$|\vec{\Phi}(s)| < |s|^{-\ell} \quad \text{for all } s \in \mathbb{C} \text{ s.t. } -\mathrm{Re}(s) > R.$$

(here we work with an arbitrary $|\cdot|$ on $\mathbb{C}^m$ and $M_{m\times m}(\mathbb{C})$).

Now, let $s_0 \in \mathbb{C}$. Given $\varepsilon > 0$, we will show that

$$\left| \vec{\Phi}(s_0) \right| < \varepsilon \quad , \quad \text{hence} \quad \phi(s) = 0 .$$

Note that $\quad \vec{\Phi}(s_0) = \mathbb{D}^N \cdot \vec{\Phi}(s_0 + N) \quad (\forall \; N \geq 0)$.

By Gelfand's formula : $\quad \lim\limits_{N \to \infty} \left| \mathbb{D}^N \right|^{1/N} = \rho(\mathbb{D}) \leq 1$.

Pick $\ell \geq 1$ and $R \gg 0$ so that $\quad |\vec{\Phi}(s)| < |s|^{-\ell} \quad \forall \; s : -\mathrm{Re}(s) > R$.

Let $N_0 > 0$ be s.t. $-\mathrm{Re}(s_0 + n) > R$ for every $n \geq N_0$. Then,

$$|\vec{\Phi}(s_0)| \leq |\mathbb{D}^n| \cdot |\vec{\Phi}(s_0+n)| \leq \frac{|\mathbb{D}^n|}{|s_0+n|^{\ell}} \longrightarrow 0 \quad \text{as } n \to \infty$$

$\square$

<u>Example</u>. $\quad \mathcal{D}(T) = T - \frac{1}{2} \quad \rightsquigarrow \quad \phi(s+1) = \frac{\phi(s)}{2}$

$\phi(s) = 2^{-s} = e^{-\ln(2)s}$ is a non-zero solution of this hgs equation. $\quad \phi(s) \sim 0$ as $\mathrm{Re}(s) \to \infty$

has no asymptotic power series as $\mathrm{Re}(s) \to -\infty$.

§3. Difference eq^n $\quad \boxed{\mathcal{D}(T) \cdot f(s) = g(s)}$; K = order of vanishing of $\mathcal{D}(T)$ at $T = 1$.

Assume : (a) $g(s)$ is hol. on some neighbourhood of ∞

and $\quad g(s) = \sum\limits_{n=K}^{\infty} g_n s^{-n-1}$ Taylor series near ∞.

(b) Let $(\mathcal{B}g)(t) = \sum_{n=k}^{\infty} g_n \frac{t^n}{n!}$ (check: radius of convergence $= \infty$)

Assume that $\mathcal{B}g$ has sub-exponential growth as $|t| \to \infty$.
(i.e. $\exists\, M, R, C > 0$ s.t. $|\mathcal{B}g(t)| < M \cdot e^{C|t|}$, $\forall\, |t| > R$)

(c) All roots of $\mathcal{D}(T) = 0$ lie on the unit circle.

Let $F(s) \in s^{-1}\mathbb{C}[[s^{-1}]]$ denote the formal soln. to $\mathcal{D}(T)\cdot f = g$ (see Lemma §1).

<u>Theorem.</u> – There exist two (meromorphic) solutions of $\mathcal{D}(T)\cdot f = g$, denoted by $f^+(s)$ and $f^-(s)$, uniquely determined by two conditions:

- $f^{\pm}(s)$ is holomorphic on $\pm \mathrm{Re}(s) \gg 0$.
- $f^{\pm}(s) \sim F(s)$ as $\pm \mathrm{Re}(s) \to \infty$.

Moreover $f^{\pm}(s)$ are holomorphic on larger domain (for any $\delta > 0$):

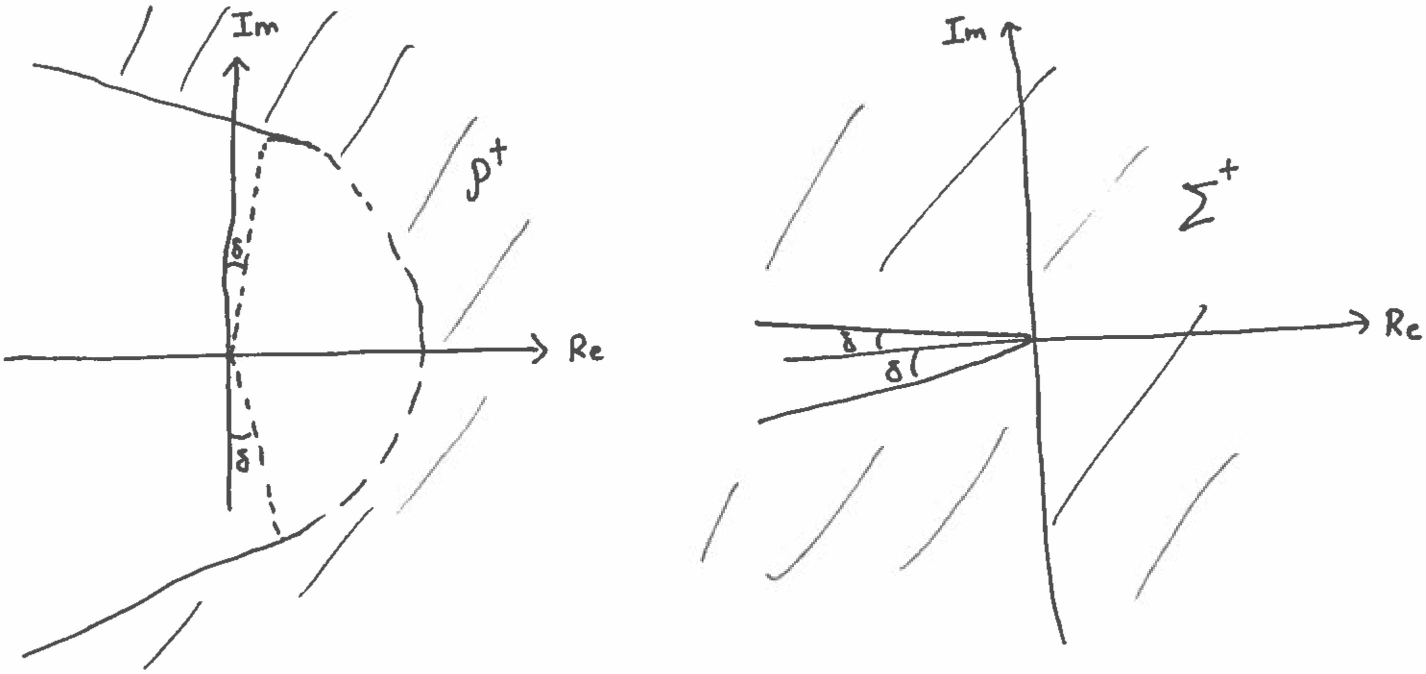
$$\mathcal{P}_\delta^+ = \bigcup_{\psi \in \left(-\frac{\pi}{2}+\delta,\, \frac{\pi}{2}-\delta\right)} \mathbb{H}_{\psi, C}$$

(recall $\mathbb{H}_{\psi,C} = \{z : \mathrm{Re}(z e^{-i\psi}) > C\}$ half plane in ψ-direction)

$(\mathcal{P}^- = -\mathcal{P}^+)$

and $f^{\pm}(s) \sim F(s)$ is valid for $s \to \infty$ in $\Sigma_\delta^{\pm}$

$$\Sigma_\delta^+ = \left\{ r e^{i\psi} : \begin{array}{l} r \in \mathbb{R}_{>0} \\ \psi \in (-\pi+\delta, \pi-\delta) \end{array} \right\} \quad ; \quad \Sigma^- = -\Sigma^+.$$



Proof.- Uniqueness is clear - by Prop §2 above.

For existence, we use Laplace transform: $f(s) = \int K(t)\, e^{-ts}\, dt$

$$(\mathcal{D}(T)\cdot f)(s) = \int K(t) \sum_{j=0}^{m} d_j\, e^{-t(s+j)}\, dt$$

$$= \int \left(K(t)\, \mathcal{D}(e^{-t}) \right) e^{-ts}\, dt$$

Comparing with $g(s) = \int (\mathcal{B}g)(t)\, e^{-ts}\, dt$, our answer is

$$f^+(s) = \int_{\ell_0} \frac{(\mathcal{B}g)(t)}{\mathcal{D}(e^{-t})} e^{-ts}\, dt \; ; \quad f^-(s) = \int_{\ell_\pi} \frac{(\mathcal{B}g)(t)}{\mathcal{D}(e^{-t})} e^{-ts}\, dt$$

(recall: $\ell_\psi = \mathbb{R}_{>0}\, e^{i\psi}$: ray along phase ψ.)

By our assumptions, $\dfrac{\mathcal{B}g(t)}{\mathcal{D}(e^{-t})}$ is holomorphic near $t=0$, has poles on the imaginary axis, and sub-exponential growth along any ℓ_ψ $\left(\psi \not\equiv \pm\frac{\pi}{2} \pmod{2\pi}\right)$.

Hence, our theorem is a consequence of Lecture 14, Lemma §2 (see also Lecture 15, Thm §3).

$\square$