Lecture 21

Asymptotic expansions of integrals

§1. Laplace method*. This method is useful for determining leading behaviour, as $x \to \infty$, of

$$I(x) = \int_a^b f(t)\, e^{-x\phi(t)}\, dt$$

The idea is that the leading contribution to $I(x)$ comes from a small neighbourhood of the minimum value of ϕ.
We begin by considering the situation when ϕ has a unique min. at some $c \in (a,b)$.

Theorem (Laplace, 1774). Assume that f is continuous on $[a,b]$, ϕ has unique min at $c \in (a,b)$ and $\phi''(c) > 0$, $f(c) \neq 0$.

Then $$\boxed{I(x) \sim f(c)\, e^{-x\phi(c)} \cdot \sqrt{\frac{2\pi}{x\,\phi''(c)}} \quad \text{as } x \to \infty}.$$

Proof. Write $$I(x) = \int_a^{c-\varepsilon} + \int_{c-\varepsilon}^{c+\varepsilon} + \int_{c+\varepsilon}^{b} f(t)\, e^{-x\phi(t)}\, dt$$

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* Pierre-Simon Laplace 1749-1827

Ex: $\int_a^{c-\varepsilon}$ and $\int_{c+\varepsilon}^{b}$ give asymptotically negligible contribution to $I(x)$, compared to $\int_{c-\varepsilon}^{c+\varepsilon}$.

Thus, $I(x) \sim \int_{c-\varepsilon}^{c+\varepsilon} f(t)\, e^{-x\phi(t)}\, dt$. Now, near c,

$\phi(t) = \phi(c) + \frac{\phi''(c)}{2}(t-c)^2 + \cdots$. By choosing ε small enough, we can replace $f(t)$ by $f(c)$ (Ex: justify this step).

$$\int_{c-\varepsilon}^{c+\varepsilon} f(c)\, e^{-x\left(\phi(c) + \frac{\phi''(c)}{2}(t-c)^2 + \cdots\right)}\, dt$$

$$= f(c)\, e^{-x\phi(c)} \int_{-\varepsilon}^{\varepsilon} e^{-x\frac{\phi''(c)}{2}\tau^2(1+\cdots)}\, d\tau \quad (\tau = t-c)$$

Ex: $$\int_{-\varepsilon}^{\varepsilon} e^{-\frac{\phi''(c)}{2}x\tau^2(1+c_1\tau+\cdots)}\, d\tau \sim \int_{-\varepsilon}^{\varepsilon} e^{-\frac{\phi''(c)}{2}x\tau^2}\, d\tau$$

Finally, for $x \gg 0$, $$\int_{-\varepsilon}^{\varepsilon} e^{-\frac{\phi''(c)}{2}x\tau^2}\, d\tau \sim \int_{-\infty}^{\infty} e^{-\frac{\phi''(c)x}{2}\tau^2}\, d\tau$$

$$= \sqrt{\frac{2\pi}{x\,\phi''(c)}} \quad \square$$

§2. Remarks and examples. –

(a) Laplace's formula remains valid in much greater generality than the one considered in the theorem above

(b) Ex. $\displaystyle\int_0^\infty e^{-Kt^n}\,dt = \frac{1}{n}\Gamma\left(\frac{1}{n}\right) K^{-1/n}$.

Use this to prove that: if $\phi'(c) = 0 = \phi''(c) = \cdots = \phi^{(p-1)}(c)$
(same assumptions on f as in Thm §1 above). $\phi^{(p)}(c) > 0$ and p is even.

then $$I(x) \sim f(c)\, e^{-x\phi(c)} \cdot \frac{2\Gamma\left(\frac{1}{p}\right)}{p}\left(\frac{p!}{x\,\phi^{(p)}(c)}\right)^{\frac{1}{p}}.$$

(c) If $f(c) = 0$ (or even infinity), the method still works to give the leading term.

§3. Critical point at the ends of (a,b). – Watson's lemma.

(same proof as the one given in Lecture 13, §3 - page 6).

Lemma. Let $f : (0,\infty) \to \mathbb{C}$ be such that $f(t)e^{-zt}$ is integrable for $\mathrm{Re}(z) > 0$. If $f(t) \sim \sum_{n=0}^{\infty} c_n t^{\alpha_n}$, where $-1 < \mathrm{Re}(\alpha_0) < \mathrm{Re}(\alpha_1) < \cdots \nearrow \infty$; as $t \to 0^+$, then

$$\int_0^\infty f(t)\,e^{-zt}\,dt \sim \sum_{n=0}^{\infty} c_n\,\Gamma(\alpha_n+1)\,z^{-\alpha_n-1} \quad \text{as } \mathrm{Re}(z) \to \infty.$$

Remark.- Watson's lemma is a refined version of Laplace method, for monotone $\phi : (a,b) \to \mathbb{R}$, with unique min. at a.

Ex: If $\phi : (0,\infty) \to (0,\infty)$ is monotonically increasing fn. and $\phi'(0) > 0$, $f : [0,\infty) \to \mathbb{C}$, then

$$\int_0^\infty f(t)\, e^{-x\phi(t)}\, dt \sim \frac{f(a)\, e^{-x\phi(a)}}{x\,\phi'(a)} \quad \text{as } x \to \infty.$$

In the case of monotonically increasing $\phi : (0,\infty) \to (0,\infty)$, we can set $w = \phi(t)$, $dw = \phi'(t)\, dt$; so

$$\int_0^\infty f(t)\, e^{-x\phi(t)}\, dt = \int_0^\infty \frac{f(\phi^{-1}(w))}{\phi'(\phi^{-1}(w))}\, e^{-xw}\, dw$$

Now we can apply Watson's lemma, provided $\frac{f(\phi^{-1}(w))}{\phi'(\phi^{-1}(w))} \sim \sum_{n=0}^\infty c_n w^{\alpha_n}$ as $w \to 0^+$ can be computed.

§4. Examples.-

(1)
$$I(x) = \int_0^{\pi/2} e^{-x \sin^2(t)}\, dt = \int_0^1 \frac{1}{2}\, w^{-1/2} (1-w)^{-1/2}\, e^{-xw}\, dw$$

$\sin^2 : (0, \frac{\pi}{2}) \xrightarrow{\sim} (0,1)$

$w = \sin^2(t)$

$dw = 2 \sin(t) \cos(t)\, dt$

$= 2\sqrt{w(1-w)}\, dt$

For w near 0, $\quad (1-w)^{-1/2} = \sum_{\ell=0}^{\infty} \binom{2\ell}{\ell} \frac{w^\ell}{4^\ell}$

So, $$I(x) = \frac{1}{2}\int_0^1 \left(\sum_{\ell=0}^{\infty} \binom{2\ell}{\ell} \frac{w^{\ell-\frac{1}{2}}}{4^\ell} \right) e^{-xw}\, dw$$

$$\sim \frac{1}{2} \sum_{\ell=0}^{\infty} \binom{2\ell}{\ell} \frac{\Gamma(\ell+\frac{1}{2})}{4^\ell}\, x^{-\ell-\frac{1}{2}} \qquad \text{as } x \to \infty.$$

$\square$

<u>Note</u>: In $\frac{1}{2}\int_0^1 w^{-1/2}(1-w)^{-1/2} e^{-xw}\, dw$, $\quad f(w) = w^{-1/2}(1-w)^{-1/2}$ has a singularity at $w=0$: the unique min of $\phi(w) = w$ on $[0,1]$.

(2) $$I(x) = \int_0^1 \sin(t)\, e^{-x\sinh^4(t)}\, dt \;; \qquad \phi(t) = \sinh(t) = \frac{e^t - e^{-t}}{2} = t + \cdots$$

$: (0,\infty) \xrightarrow{\sim} (0,\infty)$

$f(t) = \sin(t)$ vanishes at $\underbrace{t=0}_{\text{MIN of } \phi(t)}$.

$$\frac{d}{dt}\sinh(t) = \frac{e^t + e^{-t}}{2} > 0$$

<u>monotonically increasing</u>

To compute the leading term of $I(x)$, we narrow around 0, and replace $f(t) = t + \cdots$, $\quad \phi(t) = t^4 + \cdots$

$$I(x) \sim \int_0^{\varepsilon} \sin(t)\, e^{-x\sinh^4(t)}\, dt \sim \int_0^{\varepsilon} t\, e^{-xt^4}\, dt$$

$$\sim \int_0^\infty t\, e^{-xt^4}\, dt$$

$$= \int_0^\infty \frac{1}{4\sqrt{x}}\, \tau^{-1/2} e^{-\tau}\, d\tau$$

$$= \frac{\Gamma(\frac{1}{2})}{4\sqrt{x}} = \frac{1}{4}\sqrt{\frac{\pi}{x}}.$$

Set $\tau = x t^4$

$$d\tau = 4x t^3 dt$$
$$= 4x \left(\frac{\tau}{x}\right)^{3/4} dt$$

So, $t\, dt = \left(\frac{\tau}{x}\right)^{\frac{1}{4}} (4x)^{-1} \left(\frac{\tau}{x}\right)^{-3/4}$

$$= \frac{1}{4x}\left(\frac{\tau}{x}\right)^{-1/2} = \frac{1}{4\sqrt{x}} \cdot \tau^{-1/2}$$

So, $\displaystyle\int_0^1 \sin(t)\, e^{-x \sinh^4(t)}\, dt \sim \frac{1}{4}\sqrt{\frac{\pi}{x}}$ as $x \to \infty$

(3) $I(x) = \displaystyle\int_0^\infty \frac{e^{-x\cosh(t)}}{\sqrt{\sinh(t)}}\, dt$. Leading behaviour $\cosh(t) = 1 + \frac{t^2}{2} + \cdots$

$\sinh(t) = t - \frac{t^3}{3!} + \cdots$

$$\sim \int_0^\infty \frac{e^{-x(1+\frac{t^2}{2})}}{\sqrt{t}}\, dt = e^{-x} \int_0^\infty e^{-x\frac{t^2}{2}} \cdot t^{-1/2}\, dt$$

$$= e^{-x} \int_0^\infty e^{-\tau} x^{-1/4} (2\tau)^{-3/4}\, d\tau$$

$$= e^{-x} \cdot \frac{x^{-1/4}}{8^{1/4}}\, \Gamma\left(\tfrac{1}{4}\right)$$

$\left(\tau = \frac{xt^2}{2} \quad d\tau = x \cdot t \cdot dt \right.$

$$= \sqrt{2\tau x}\, dt$$

$t = \sqrt{\frac{2\tau}{x}}$

$$\left(t^{-1/2} dt = \left(\frac{2\tau}{x}\right)^{-1/4} (2\tau x)^{-\frac{1}{2}}\, d\tau \right)$$