Lecture 22

Recall: Laplace method for determining leading order behaviour of $I(x) = \int_a^b f(t)\, e^{-x\phi(t)}\, dt$

$$\sim f(c)\, e^{-x\phi(c)} \sqrt{\frac{2\pi}{x\,\phi''(c)}} \quad \text{as } x \to \infty$$

if $c \in (a,b)$ is the unique min of ϕ on $[a,b]$, $\phi''(c) > 0$ and $f(c) \neq 0$.

§1. To determine full asymptotic expansion, we need Watson's

Lemma

$$\left[f(t) \sim \sum_{n=0}^{\infty} c_n t^{a_n} \text{ as } t \to 0^+ \Rightarrow \int_0^{\infty} f(t)\, e^{-xt}\, dt \sim \sum_{n=0}^{\infty} c_n\, \Gamma(a_n+1)\, x^{-a_n-1} \text{ as } x \to \infty. \right]$$

Thus, in order to obtain higher order terms in Laplace's formula, we restrict ourselves to a neighbourhood of c (assume $c = 0$ for simplicity). $\phi(t) - \phi(0) = \frac{\phi''(0)}{2} t^2 + \cdots$ is 2-1 map near 0

so, it has 2 local inverses: on $(-\varepsilon, 0)$ and on $(0, \varepsilon)$

$$I(x) \sim \underbrace{\int_{-\varepsilon}^{0} f(t)\, e^{-x\phi(t)}\, dt}_{I_-(x)} + \underbrace{\int_{0}^{\varepsilon} f(t)\, e^{-x\phi(t)}\, dt}_{I_+(x)}$$

$$I_+(x) = \int_0^{\varepsilon} f(t)\, e^{-x(\phi(t)-\phi(0))} \cdot e^{-x\phi(0)}\, dt$$

$w = \phi(t) - \phi(0)$ can be inverted

$$= \frac{\phi''(0)}{2} t^2 + \cdots$$

$$t = a(w) + \sqrt{w}\, b(w) = \sum_{n=1}^{\infty} a_n w^n + \sum_{m=0}^{\infty} b_m w^{m+\frac{1}{2}}$$

say, $t = h(w)$

$$\left(b_0^2 = \frac{2}{\phi''(0)} \right)$$

$$dt = h'(w)\, dw$$

So $$I_+(x) = \int_0^{\varepsilon' = \phi(\varepsilon)-\phi(0)} \underbrace{f(h(w)) \cdot h'(w)}_{} \, e^{-xw}\, dw$$

as $w \to 0$, $\left(f(0) + w^{1/2}(\cdots)\right)\left(\frac{b_0}{2} w^{-1/2} + \cdots\right)$

Watson's lemma applies. The problem becomes that of inverting a power series.

$$I_+(x) \sim f(0) \frac{1}{\sqrt{2\phi''(0)}} \Gamma\left(\tfrac{1}{2}\right) x^{-\frac{1}{2}} \quad \text{as} \quad x \to \infty.$$

Repeating the same argument for $I_-(x) = \int_{-\varepsilon}^{0} f(t)\, e^{-x\phi(t)}\, dt$

gives $$I(x) \sim f(0) \sqrt{\frac{2}{x\phi''(c)}} \cdot \Gamma\left(\tfrac{1}{2}\right).$$

Thus, we can obtain Laplace's formula from Watson's lemma.

§2. Stirling Series again.

Recall $$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$$

$$= \cancel{x}^{x+1} \int_0^\infty s^x e^{-xs}\,ds \qquad \begin{pmatrix} \text{set} & t = x\cdot s \\ & dt = x\cdot ds \end{pmatrix}$$

$$\frac{\Gamma(x+1)}{x^{x+1}} = \int_0^\infty e^{-x(s-\ln(s))}\,ds$$

$\phi(s) = s - \ln(s)$ has unique min. at $s=1$

$$\left(\phi'(s) = 1 - \frac{1}{s} \quad \begin{matrix} <0 & \text{for } s\in(0,1) \\ >0 & \text{for } s\in(1,\infty) \end{matrix}\right) \qquad \begin{matrix} \phi(1) = 1 \\ (\phi''(1) = 1). \end{matrix}$$

$$I(x) = \int_0^\infty e^{-x(s-\ln(s))}\,ds \sim \underbrace{\int_{1-\varepsilon}^{1}}_{-I_-(x)} + \underbrace{\int_1^{1+\varepsilon} e^{-x(s-\ln(s))}\,ds}_{I_+(x)}$$

Focus on $I_+(x)$: $\qquad t = s-1$

$$I_+(x) = \int_0^\varepsilon e^{-x(t+1-\ln(t+1))}\,dt$$

$$= e^{-x} \int_0^\varepsilon e^{-x(t-\ln(t+1))}\,dt$$

$$w = t - \ln(1+t) = \frac{t^2}{2} - \frac{t^3}{3} + \frac{t^4}{4} - \cdots$$

can be (locally) inverted to $\quad t = h(w) = \sum_{n=1}^{\infty} a_n w^n + \sum_{m=0}^{\infty} b_m w^{m+\frac{1}{2}} \quad (b_0 = \sqrt{2})$

$$dt = h'(w)\,dw = \left(\sum_{n=1}^{\infty} n a_n w^{n-1} + \sum_{m=0}^{\infty} \left(m+\tfrac{1}{2}\right) b_m w^{m-\frac{1}{2}} \right) dw$$

$$I_+(x) = e^{-x} \int_0^{\varepsilon'} h'(w)\, e^{-xw}\, dw$$

$$\sim e^{-x} \left(\sum_{m=0}^{\infty} \left(m+\tfrac{1}{2}\right) b_m \Gamma\left(m+\tfrac{1}{2}\right) x^{-m-\frac{1}{2}} + \sum_{n=1}^{\infty} n a_n \cdot (n-1)!\, x^{-n} \right)$$

Same analysis as above gives $I_-(x)$ from $t = \tilde{h}(w) = \sum a_n w^n - \sum b_m w^{m+\frac{1}{2}}$

(check)

So, $$I(x) \sim 2 e^{-x} \left(\sum_{m=0}^{\infty} b_m \Gamma\left(m+\tfrac{3}{2}\right) x^{-m-1} \right)$$

Exercise: Solve for first few terms $b_0, a_1, b_1, a_2, b_2, \ldots$

Hint: $\quad h'(w) = \dfrac{dt}{dw} = \dfrac{1 + h(w)}{h(w)}$; write $h(w) = a(w) + \sqrt{w}\, b(w)$

$$\left(a(w) + \sqrt{w}\, b(w)\right) \left(a'(w) + \tfrac{1}{2} w^{-\frac{1}{2}} b(w) + \sqrt{w}\, b'(w) \right)$$

$$= 1 + a(w) + \sqrt{w}\, b(w)$$

and compare coefficients of $w^0, w^{\frac{1}{2}}, w^1, w^{\frac{3}{2}}, \ldots$

§3. Stationary phase / steepest descent ... method.

Example $$I(\lambda) = \int_0^1 \frac{e^{-i\lambda t^2}}{1+t}\,dt$$ as $\lambda \to \infty$ (assume $\lambda \in \mathbb{R}_{>0}$).

Issue: $\left| e^{-i\lambda t^2} \right| = 1$ for $t \in (0,1)$.

Consider the function $\phi(t) = i\,t^2$. $(t \in \mathbb{C})$ More traditionally

$$\phi(z) = i\,z^2 = i\,(x^2 - y^2 + 2xyi) = -2xy + i\,(x^2 - y^2)$$

$$\left| e^{-\lambda\phi(z)} \right| = e^{-\operatorname{Re}(\lambda\phi(z))} = e^{\lambda(2xy)}$$

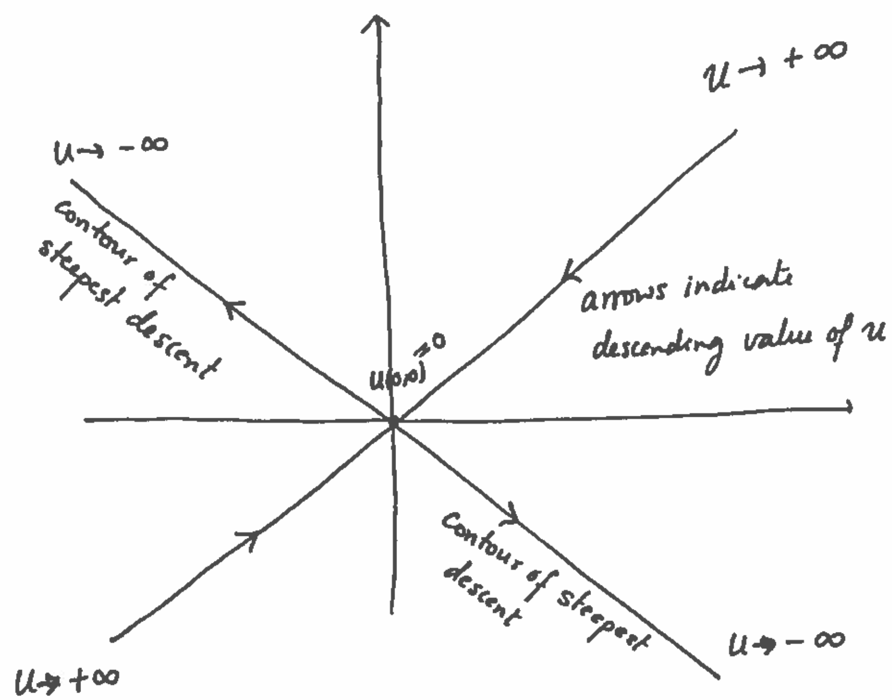
$\to$ If we are integrating along a contour $C : z_0 \longrightarrow \infty$ along which $2\,x_0 y_0 > 2\,xy$ $\forall\, z = x+iy$, then Laplace's method applies and $\displaystyle\int_C \frac{e^{-i\lambda t^2}}{1+t}\,dt$ can be computed from the behaviour of integrand near z_0. ["Steepest descent contour"]

Rewriting $$I(\lambda) = \int_0^1 \frac{e^{\lambda(\overbrace{2xy - i\,(x^2-y^2)}^{\psi(z) = -iz^2})}}{1 + \underbrace{(x+iy)}_{z}}\,dz$$

For the endpoint 0, $\psi(0) = 0$

[Recall: for $\psi(z) = u(x,y) + i\,v(x,y)$, the curve of steepest ascent/descent for $u(x,y)$ are the level curves of $v(x,y)$.]

In our case $v(x,y) = y^2 - x^2$. Its level curves through $(0,0)$ are $y = \pm x$



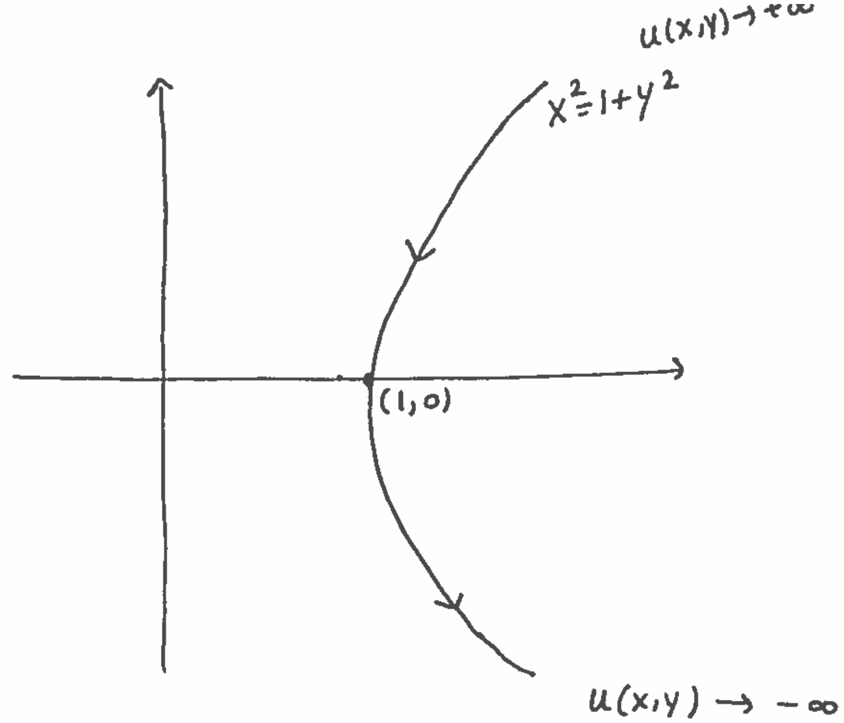
$u(x,y) = 2xy$

level curves $y^2 - x^2 = 0$

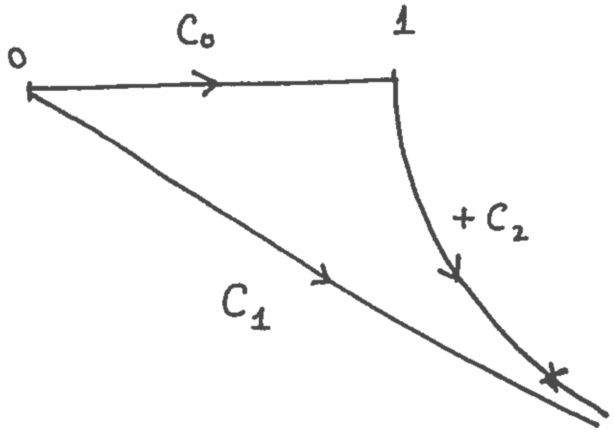
For the endpoint 1, $\psi(1) = -i$

Level curves for $y^2 - x^2 = -1$, $x^2 = 1 + y^2$ are hyperbolas

Now, we argue by principle of contour deformation to replace $[0,1]$ by contours of steepest descent.



level curve $v(x,y)$ through $(1,0)$



$$\int_{C_0} = \int_{C_1} - \int_{C_2}$$

Easy exercise – justify this.

<u>Analysis of C_1</u> $= e^{-i\frac{\pi}{4}} \cdot t \quad (0 < t < \infty)$ set $z = e^{-i\frac{\pi}{4}} t$, so $z^2 = -i t^2$

$-i z^2 = -t^2$

$$I_1(\lambda) = \int_0^\infty \frac{e^{-\lambda t^2}}{1 + e^{-i\frac{\pi}{4}} t} \, e^{-i\frac{\pi}{4}} \, dt \qquad \text{(back in familiar form)}$$

<u>Analysis of C_2</u>

$$I_2(\lambda) = e^{-i\lambda} \int_0^\infty \frac{e^{-\lambda p}}{1 + Z(p)} \, Z'(p) \, dp \qquad \left(\frac{Z'(p)}{1+Z(p)} = \frac{i}{2}(1+ip)^{-1/2} \cdot \left(1 + (1+ip)^{1/2}\right)^{-1} \right.$$