Recall: we were studying the problem of determining $\lambda \to \infty$ ($\lambda \in \mathbb{R}$) behaviour of

$$I(\lambda) = \int_C f(z)\, e^{\lambda \psi(z)}\, dz .$$

Here, C is a (finite or infinite) ~~contour~~ curve/path in $\Omega \subset \mathbb{C}$; $f, \psi : \Omega \to \mathbb{C}$ are holomorphic.

Note: if $\psi(x+yi) = u(x,y) + i\,v(x,y)$, then

$|e^{\lambda\psi(z)}| = e^{\lambda \cdot u(x,y)}$. Therefore, if C has an endpoint, say z_0, so that $u(x_0, y_0) = \text{Max}\{u(x,y) : (x,y) \in C\}$, then asymptotic expansion of $I(\lambda)$, as $\lambda \to \infty$, can be obtained via Laplace's method + Watson's lemma.

§1. Definition. Let $C : (a,b) \to \Omega$ be a path in Ω ; $\psi : \Omega \to \mathbb{C}$ holomorphic.

We say C is a path of ascent (resp. descent) for $u(x,y) = \text{Re}\,\psi(x+iy)$, if the composition $(a,b) \xrightarrow{C} \Omega \xrightarrow{u} \mathbb{R}$ is a monotonically increasing (resp. decreasing) function.

C is a path of steepest ascent (resp. descent) if

$$\vec{C}'(t) = k\,\vec{\nabla} u(C(t)) \quad \forall\, t \in (a,b) \text{ where } C \text{ is differentiable,}$$

for some $k > 0$ (resp. $k < 0$).

(i.e. tangent line to C at a point p is in the same (resp. opposite) direction as $\vec{\nabla} u$.)

Prop. (see Lecture 1)

Let $z_0 \in \Omega$ be such that $\psi'(z_0) \neq 0$. Let $\psi(z_0) = A + Bi$. If C is a path of steepest ascent (or descent) passing through z_0, then $v(x,y) = B$ $\forall (x,y)$ on C.

Proof. If C is given by $\{(x(t), y(t)) : a \le t \le b\}$, then

$$\frac{d}{dt} v(x(t), y(t)) = v_x \cdot x'(t) + v_y \cdot y'(t)$$

$$= k(v_x u_x + v_y u_y) \qquad (\langle x'(t), y'(t)\rangle = k \vec{\nabla} u)$$

$$= k(v_x u_x + (-u_x) v_x) \qquad \text{(by Cauchy-Riemann equations)}$$

$$= 0. \qquad \square$$

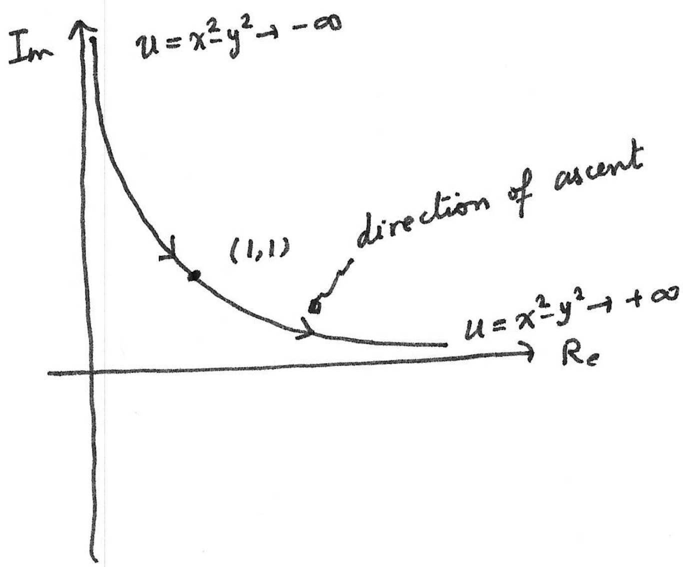
Example. $\psi(z) = z^2 = x^2 - y^2 + 2xy\,i$; $\psi'(z) = 2z$.

$u = x^2 - y^2$; $v = 2xy$.

If $z_0 \neq 0$, say for instance $z_0 = 1 + i$ $(u = 0 ; v = 2)$

then along $2xy = 2$ (or $xy = 1$ hyperbola through $(1,1)$)

$x^2 - y^2$ varies fastest from $-\infty$ to ∞



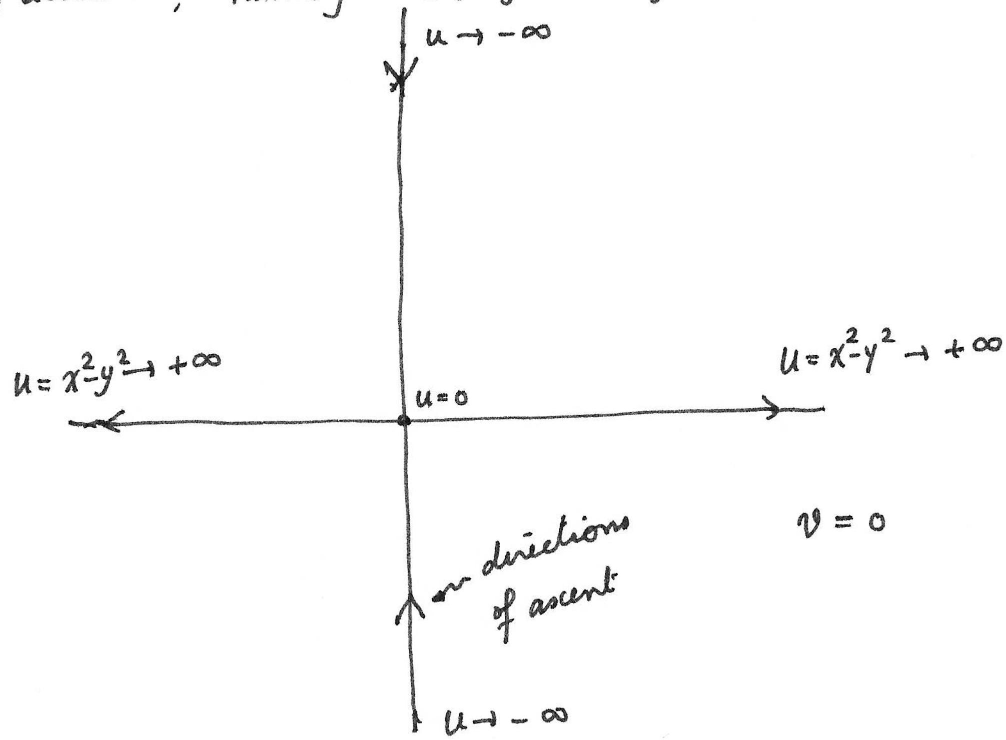
Ex. let ~~γ(t)~~ $C = \{(\frac{1}{t}, t) : t \ge 1\}$

Use Laplace/Watson method to determine the leading behaviour of

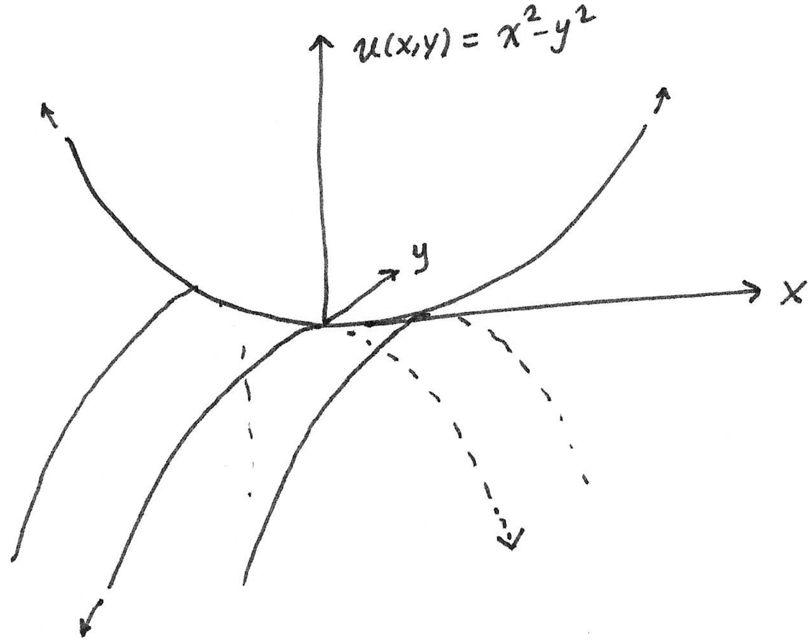
$$\int_C \frac{e^{\lambda z^2}}{1 + z}\, dz .$$

Example cntd. ($\psi(z) = z^2$). At $z_0 = 0$, there are 4 directions for steepest ascent / descent, namely $v(x,y) = 2xy = 0$

Thus the graph $\{(x,y,x^2-y^2) : x,y \in \mathbb{R}\}$ has a <u>saddle point</u> at 0



graph of $x^2 - y^2$:

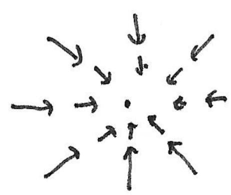


§2. Remarks. - For $\psi(z) = u(x,y) + i\,v(x,y)$, holomorphic, Cauchy-Riemann equations imply $\vec{\nabla} u \cdot \vec{\nabla} v = 0$.

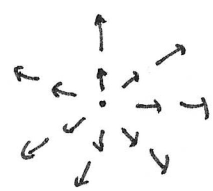
Moreover, $\vec{\nabla} u\,(x_0, y_0) = 0 \iff \vec{\nabla} v\,(x_0, y_0) = 0 \iff \psi'(z_0) = 0$.

<u>Defn.</u> z_0 is a critical point of ψ if $\psi'(z_0) = 0$.

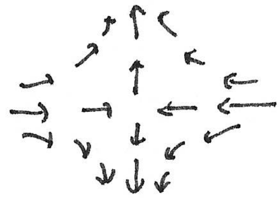
In general - for a $\mathbb{R}$-valued function $f(x,y)$, $\vec{\nabla} f$ has the following possibilities near a critical point:



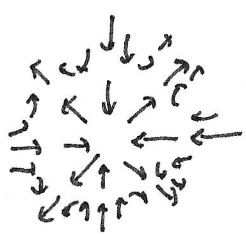
Local Max.



Local Min



Saddle point (order 1)



Sadde point (order 2)

The point of the previous proposition is that for f harmonic, critical points can only be saddle points.

Path of steepest ascent is determined by

$$x'(t) = u_x(x(t), y(t))$$
$$y'(t) = u_y(x(t), y(t))$$

Previous proposition solves this system "implicitly" as

$$v(x(t), y(t)) = \text{constant}.$$

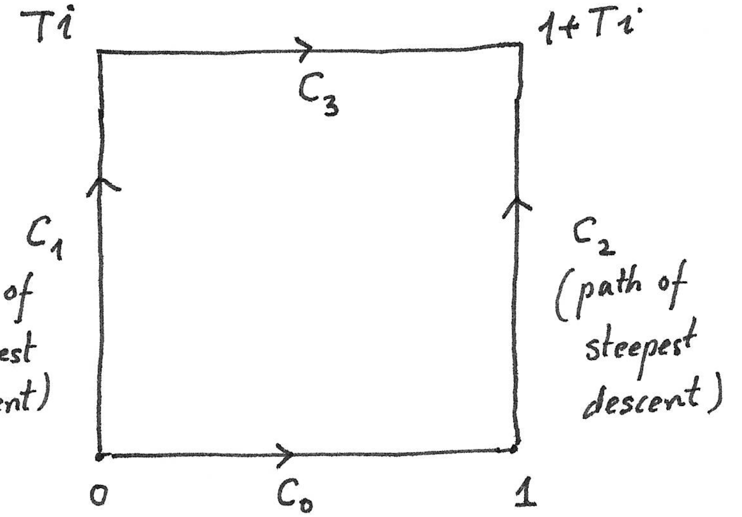
(where v is the harmonic conjugate of u)

§3. Method of steepest descent - example (even more basic)

$$I(\lambda) = \int_0^1 \ln(t)\, e^{i\lambda t}\, dt .$$

$\psi(z) = iz$, so $u(x,y) = -y$, $v(x,y) = x$. The path of steepest descent of $u(x,y)$ from any (x_0, y_0) is the vertical line (upwards) $x = x_0$.

$\psi'(z) = i \neq 0$.



By Cauchy's theorem,

$$I(\lambda) = \int_{C_1} - \int_{C_2} + \int_{C_3} \log(z)\, e^{i\lambda z}\, dz$$

Claim.

$$\lim_{T\to\infty} \int_{Ti}^{1+Ti} \log(z)\, e^{iz\lambda}\, dz = 0 .$$

Pf.

$$\int_{Ti}^{1+Ti} \log(z)\, e^{iz\lambda}\, dz = \int_0^1 \log(u+Ti)\, e^{i(u+Ti)\lambda}\, du \qquad (z = u + Ti)$$

$$= e^{-\lambda T} \int_0^1 \log(u+Ti)\, e^{iu\lambda}\, du$$

Now $\log(u+Ti) = \frac{1}{2}\ln(u^2+T^2) + i\arctan\left(\frac{T}{u}\right)$ $(0<u<1)$

$$\Rightarrow \quad |\log(u+Ti)|^2 < \frac{1}{4}\left(\ln(1+T^2)\right)^2 + \frac{\pi^2}{4}.$$

So, $$\left|\int_{Ti}^{1+Ti} \log(z)\, e^{iz\lambda}\, dz\right| \leq e^{-\lambda T}\left(\frac{1}{4}\left(\ln(1+T^2)\right)^2 + \frac{\pi^2}{4}\right)^{\frac{1}{2}}$$

$\to 0$ as $T\to\infty$. □

So, $$I(\lambda) = \lim_{T\to\infty} \int_0^{Ti} - \int_1^{1+Ti} \log(z)\, e^{iz\lambda}\, dz$$

<u>Calculation along C_1</u> ; $z = it$,

$$I_1(\lambda) = \int_0^\infty \log(it)\, e^{-t\lambda}\, i\, dt$$

$$= i\int_0^\infty \left(\ln(t) + i\frac{\pi}{2}\right) e^{-t\lambda}\, dt = -\frac{\pi}{2}\frac{1}{\lambda} + i\int_0^\infty \ln(t)\, e^{-t\lambda}\, dt.$$

$$\int_0^\infty \ln(t)\, e^{-t\lambda}\, dt = \int_0^\infty \ln\left(\frac{u}{\lambda}\right) e^{-u}\frac{du}{\lambda} = \frac{1}{\lambda}\left(\int_0^\infty \ln(u)\, e^{-u}\, du - \ln(\lambda)\int_0^\infty e^{-u}\, du\right)$$

$(u = t\lambda)$ $$= \frac{-\gamma - \ln(\lambda)}{\lambda}$$

Hence, $I_1(\lambda) = \frac{-\pi}{2\lambda} - i\,\frac{\gamma + \ln(\lambda)}{\lambda}$ (This is exact!)

<u>Calculation along C_2.</u> $z = 1 + it$

$$I_2(\lambda) = \int_0^\infty \log(1+it)\, e^{i(1+it)\lambda}\, i\, dt$$

$$= i \cdot e^{i\lambda} \int_0^\infty \boxed{\log(1+it)}\, e^{-t\lambda}\, dt$$

$$\hookrightarrow \sum_{n=1}^\infty (-1)^{n-1} \frac{(it)^n}{n}$$

By Watson's lemma: $I_2(\lambda) \sim i e^{i\lambda} \left(\sum_{n=1}^\infty (-1)^{n-1} i^n \cdot (n-1)!\, \lambda^{-n-1} \right)$

as $\lambda \to \infty$.

Hence,

$$I(\lambda) \sim \frac{-\pi}{2\lambda} - i\,\frac{\gamma + \ln(\lambda)}{\lambda} - i e^{i\lambda} \left(\sum_{n=1}^\infty (-1)^{n-1} i^n (n-1)!\, \lambda^{-n-1} \right)$$

□