Lecture 24

Airy differential equation

§0. The following 2^{nd} order, homogeneous, differential equation was discovered by G.B. Airy (1801 – 1892), around 1840, in his work on light intensity in a neighbourhood of a caustic.

$$\boxed{f''(z) = z \cdot f(z)} \qquad \text{Airy's diff'l eq}^{\underline{n}}.$$

Formally, near 0, one can solve for $f(z) = \sum_{n=0}^{\infty} a_n z^n$:

$f'' = z \cdot f$ becomes the following recurrence relation:

$$\boxed{n(n-1)\, a_n = a_{n-3}} \qquad n \geq 2, \ (a_{-1} = 0).$$

$(a_0, a_1 \in \mathbb{C}$ arbitrary$)$

Which turns into

$$a_{3k+2} = 0 \qquad \forall\, k \geq 0$$

$$a_{3k} = \frac{a_0}{(3k)(3k-1)\ (3k-3)(3k-4) \cdots (3)(2)} \qquad \forall\, k \geq 0$$

$$a_{3k+1} = \frac{a_1}{(3k+1)(3k)\ (3k-2)(3k-3) \cdots (4)(3)} \qquad \forall\, k \geq 0$$

Giving 2 fundamental solns: $f_1(z) = \sum_{l=0}^{\infty} (3l-2) \cdots 4(1) \dfrac{z^{3l}}{(3l)!} \quad \begin{bmatrix} a_0 = 1 \\ a_1 = 0 \end{bmatrix}$

$$f_2(z) = \sum_{l=0}^{\infty} \frac{z^{3l+1}}{(3l+1)!} (3l-1)(3l-4) \cdots (2).$$

§1. Integral form for solutions of Airy equation.

(Heuristic derivation) $$f(z) = \int_C F(s)\, e^{zs}\, ds$$ (F and C to be found.)

$$f''(z) = \int_C s^2 F(s)\, e^{zs}\, ds \quad , \text{ and } \quad z f(z) = \int_C z F(s) \cdot e^{zs}\, ds$$

$$z f(z) \underset{\substack{\text{(integration} \\ \text{by parts)}}}{=} \Big[F(s) \cdot e^{zs}\Big]_C - \int_C F'(s)\, e^{zs}\, ds$$

difference between the values of $F(s) \cdot e^{zs}$, at $s =$ end points of C.

Assume C is such that F(s) vanishes at the end points of C (once we find F, we will have to pick C satisfying this condition).

$$\Rightarrow \quad z f(z) = -\int_C F'(s)\, e^{zs}\, ds \ .$$ So, $\int_C F(s) e^{zs} ds$ solves Airy eq$^{\underline{n}}$ if $s^2 F(s) + F'(s) = 0$, i.e.

$$\frac{F'}{F} = -s^2 \quad \Rightarrow \quad F = \exp\left(-\int s^2 ds\right)$$

$$F(s) = e^{-s^3/3}.$$

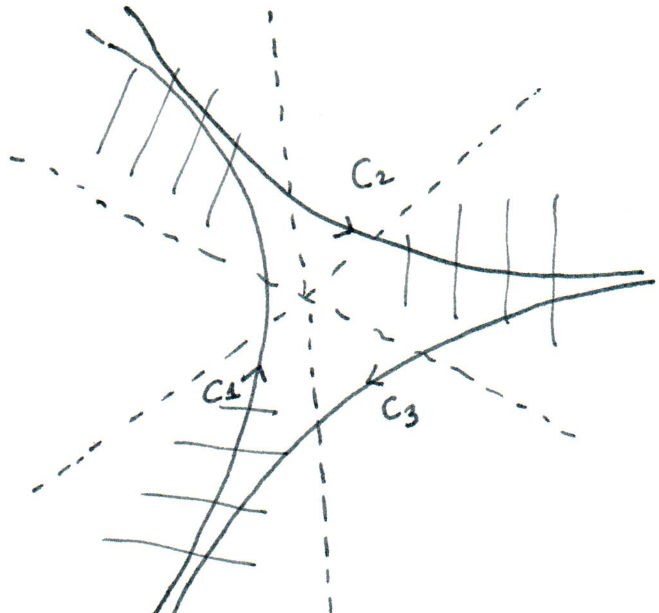
Having determined F(s), we have to choose C so that

$\underbrace{e^{-s^3/3} \to 0}$ as $s \to$ end points of C.

this is same as $\mathrm{Re}(s^3) \to +\infty$, which happens when

$$\arg(s) \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \cup \left(\frac{\pi}{2}, \frac{5\pi}{6}\right) \cup \left(\frac{7\pi}{6}, \frac{3\pi}{2}\right)$$

So, let C_1, C_2, C_3 be three infinite paths joining points at ∞ in the three region found above.



Schematically written as

$$\begin{tikzcd}
e^{i\frac{4\pi}{3}}\infty \arrow[rr, "C_1"] & & e^{i\frac{2\pi}{3}}\cdot\infty \arrow[dl, "C_2"] \\
& e^{i0}\cdot\infty \arrow[ul, "C_3"] &
\end{tikzcd}$$

. Let $f_j(z) = \int_{C_j} e^{sz - \frac{s^3}{3}}\, ds$

Ex. Verify that the integral converges uniformly for cpct subsets of $\mathbb{C}$. And $f_1(z) + f_2(z) + f_3(z) = 0 \ \forall z$.

(i.e. given $K \subset \mathbb{C}$ cpct ; $\exists R \gg 0$ s.t.
$\varepsilon > 0$

$$\left| \int_{\substack{\gamma_j(+R', +\infty) \\ \text{or} \\ \gamma_j(-\infty, -R')}} e^{sz - s^3/3}\, ds \right| < \varepsilon \quad \forall z \in K, \ R' > R.)$$

$C_j = \gamma_j : (-\infty, \infty) \to \mathbb{C}$

§2. Definition :

$$\boxed{\mathrm{Ai}(z) = \frac{1}{2\pi i} \int_{C_1} e^{sz - \frac{s^3}{3}}\, ds}$$

The factor of $2\pi i$ is for normalization, and to make sure $\mathrm{Ai}(x) \in \mathbb{R}$ for $x \in \mathbb{R}$.

<u>Ex.</u> For $x \in \mathbb{R}$, $\mathrm{Ai}(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \cos\left(\frac{\tau^3}{3} + \tau x\right) d\tau$

is $\mathbb{R}$-valued.

[Hint: justify changing C_1 to $i\mathbb{R}$ and set $s = i\tau$ $(-\infty < \tau < \infty)$

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\tau x + i\frac{\tau^3}{3}}\, d\tau$$
]

§3. <u>Asymptotic analysis of $\mathrm{Ai}(z)$, for $z \in \mathbb{R}_{>0}$, $z \to \infty$.</u>

Change to (λ, t) variables, related to (z, s) via $\begin{array}{l} zs = \lambda t \\ s^3 = \lambda \cdot t^3 \end{array}$

So, $\lambda = z^{3/2}$

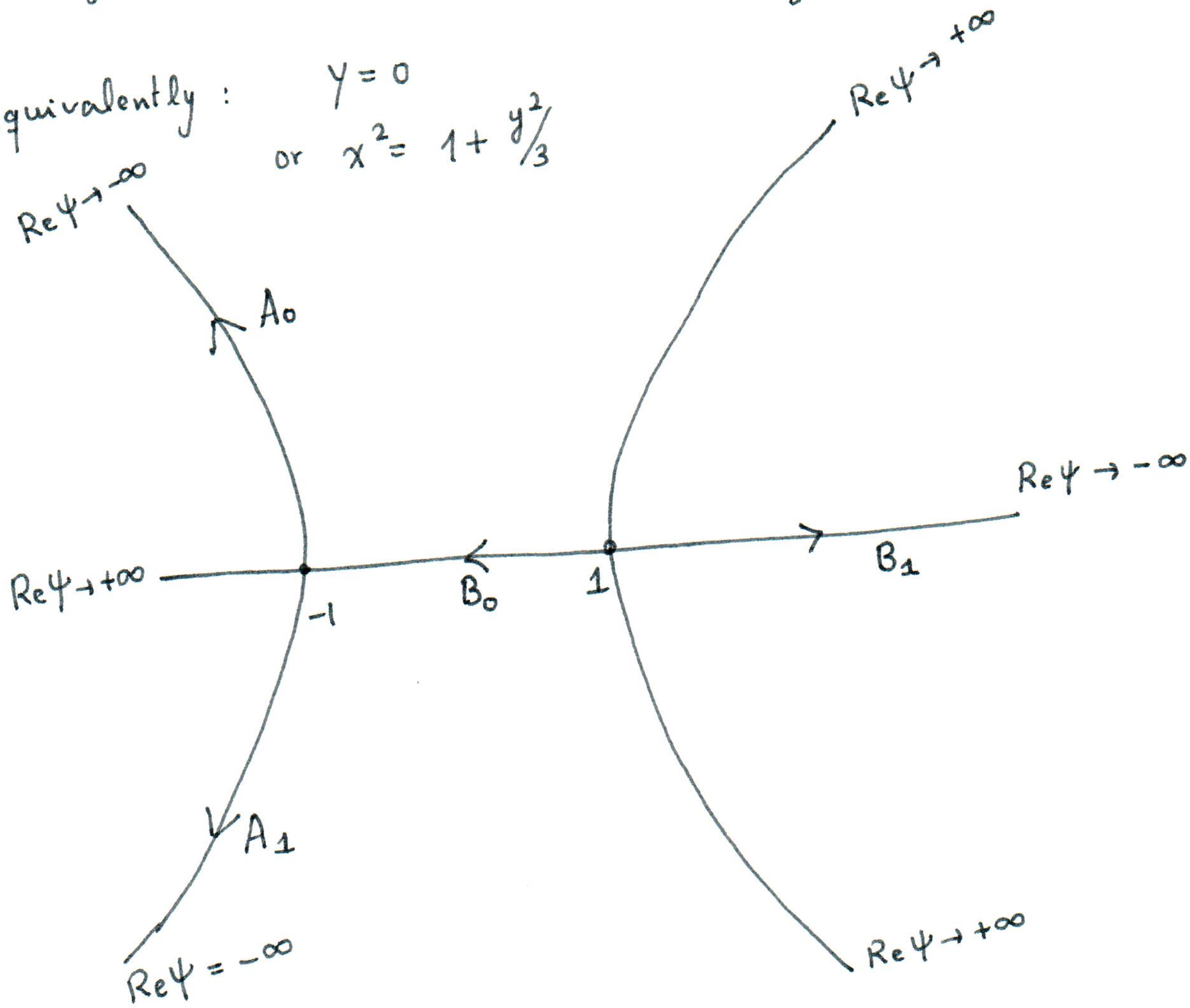
$t = z^{-1/2} s$ ⟵ [Note: this does not affect C_1 since $z \in \mathbb{R}_{>0}$, it is mere stretching]

$$\mathrm{Ai}(z) = I_1(\lambda) = \frac{\lambda^{1/3}}{2\pi i} \int_{C_1} e^{\lambda(t - t^3/3)}\, dt .$$

Let $\psi(z) = z - \frac{z^3}{3}$. (Two critical points $z = \pm 1$).

Steepest descent for $\mathrm{Re}(\psi(z))$ occurs along $\mathrm{Im}\,\psi(z) = 0$
(through -1 & 1) i.e. $y + \frac{y^3}{3} - x^2 y = 0$

equivalently: $y = 0$
or $x^2 = 1 + y^2/3$



Our contours C_1, C_2, C_3 used to define

$$f_j(z) = \frac{1}{2\pi i} \int_{C_j} e^{sz - s^3/3}\, ds \qquad (f_1 = Ai)$$

can be chosen to be

$$C_1 = -A_1 + A_0$$
$$C_2 = -A_0 - B_0 + B_1$$
$$C_3 = -B_1 + B_0 + A_1$$

Now the leading term of $f_j(z)$ (as $z \to \infty$, $z \in \mathbb{R}$) can be easily computed, using Laplace's formula :

$$f_1(x) = Ai(x) \sim \frac{x^{-1/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3} x^{3/2}\right)$$

Similarly

$$I_2(\lambda) \sim \frac{\lambda^{-1/6}}{2i\sqrt{\pi}} \exp\left(\frac{2\lambda}{3}\right)$$

$$I_3(\lambda) \sim -\frac{\lambda^{-1/6}}{2i\sqrt{\pi}} \exp\left(\frac{2}{3}\lambda\right)$$