Lecture 25

Recall: last time we started studying Airy's differential equation:

$$\boxed{f''(z) = z \cdot f(z)}$$

and introduced following solns:

Near 0:

$$F_1(z) = \sum_{\ell=0}^{\infty} \left(\prod_{j=1}^{\ell} (3j-2) \right) \frac{z^{3\ell}}{(3\ell)!}$$

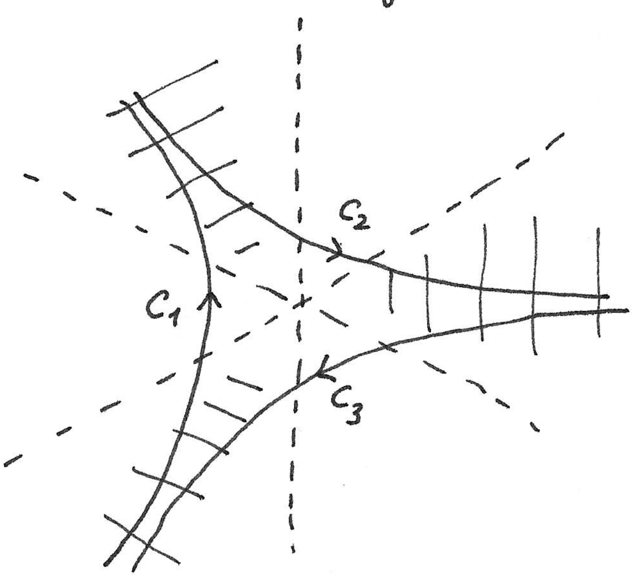
$$F_2(z) = \sum_{\ell=0}^{\infty} \left(\prod_{j=1}^{\ell} (3j-1) \right) \frac{z^{3\ell+1}}{(3\ell+1)!}$$

both have infinite radius of convergence (e.g. by d'Alembert's ratio test)

Global solns:

$$f_j(z) = \frac{1}{2\pi i} \int_{C_j} e^{sz - \frac{s^3}{3}} \, ds$$

; where $j = 1, 2, 3$, and paths of integration are given below:



Airy function:

$$Ai(z) = f_1(z)$$

Ex: Take C_1: $s = -1 + it$

$ds = i\,dt$

$$Ai(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(-1+it)z - \frac{(-1+it)^3}{3}} dt$$

$$= \frac{e^{-z+\frac{1}{3}}}{2\pi} \int_{-\infty}^{\infty} e^{itz - t^2 + i\left(-t + \frac{t^3}{3}\right)} dt$$

Assume $z = x \in \mathbb{R}$. Then, we get

$$Ai(x) = \frac{e^{-x+\frac{1}{3}}}{2\pi} \int_{-\infty}^{\infty} e^{-t^2} \left(\cos\left(tx - t + \frac{t^3}{3}\right) + i \underbrace{\sin\left(tx - t + \frac{t^3}{3}\right)}_{\text{odd fn. of } t} \right) dt$$

Hence, $Ai(x) \in \mathbb{R}, \ \forall \ x \in \mathbb{R}$;

$$Ai(x) = \frac{e^{-x+\frac{1}{3}}}{2\pi} \int_{-\infty}^{\infty} e^{-t^2} \cos\left(tx - t + \frac{t^3}{3}\right) dt.$$

§1. Asymptotic expansion of $Ai(z)$, as $z \to \infty$, $\arg(z) = \theta$ fixed.

Write $z = |z| e^{i\theta}$, $\quad Ai(z) = \frac{1}{2\pi i} \int_{C_1} e^{|z| s e^{i\theta} - \frac{s^3}{3}} ds$

Change of variables: (λ, t) so that $\quad s^3 = \lambda t^3$, $\quad |z| s = \lambda t$ $\quad$ So, $\lambda = |z|^{+\frac{3}{2}} \in \mathbb{R}_{>0}$, $\quad ds = \lambda^{\frac{1}{3}} dt$, $\quad s = \lambda^{\frac{1}{3}} t$

$$I(\lambda) = \frac{\lambda^{\frac{1}{3}}}{2\pi i} \int_{C_1} \exp\left(\lambda\left(t e^{i\theta} - \frac{t^3}{3}\right)\right) dt$$

This is in familiar Laplace-type form with

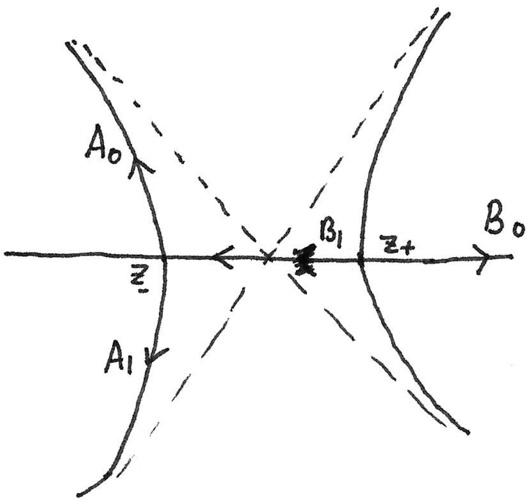
$$\psi(t) = t e^{i\theta} - \frac{t^3}{3} \qquad \psi'(t) = \alpha - t^2 \rightsquigarrow z_{\pm} = \pm\sqrt{\alpha} = \pm e^{i\theta/2}.$$

2 Critical points

$$(\text{write } \alpha = e^{i\theta} \in S^1) \qquad \psi''(t) = -2t.$$

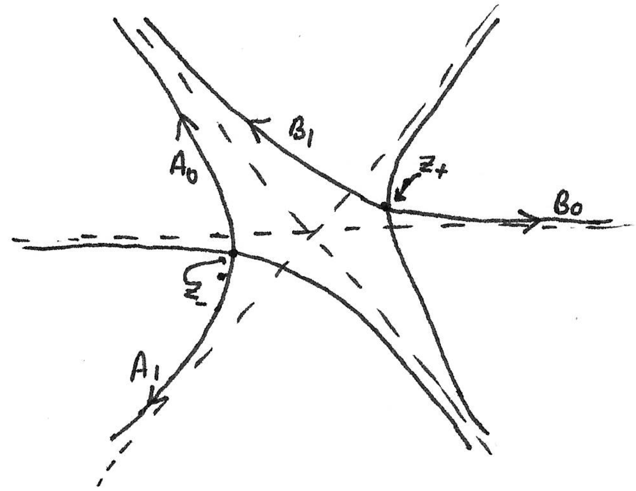
$$z_\pm = \pm e^{i\theta/2} \;;\quad \psi(z_\pm) = \pm \frac{2}{3} e^{i\frac{3\theta}{2}} \;;\quad v(x,y) = \mathrm{Im}\,\psi = x\sin\theta + y\cos\theta + \frac{y^3}{3} - x^2 y$$

§2. Plots of $v(x,y) = \pm\frac{2}{3}\sin\left(\frac{3\theta}{2}\right)$ as θ varies

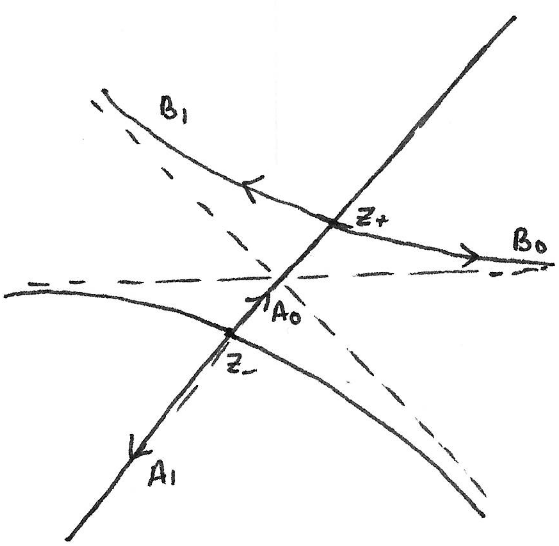


$\theta = 0$

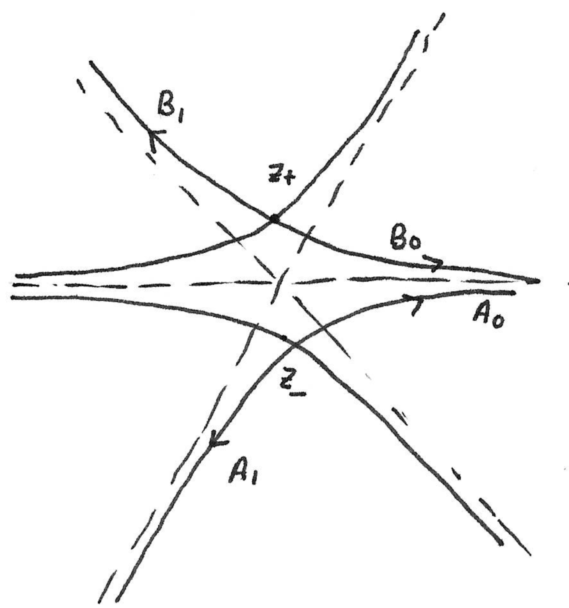
← direction of steepest descent of $u = \mathrm{Re}\,\psi$.



$0 < \theta < \frac{2\pi}{3}$



$\theta = \frac{2\pi}{3}$



$\frac{2\pi}{3} < \theta \leqslant \pi$.

Conclusion: for $I_1(\lambda) = \dfrac{\lambda^{1/3}}{2\pi i} \displaystyle\int_{C_1} \exp\left(\lambda\left(te^{i\theta} - \frac{t^3}{3}\right)\right) dt$,

C_1 can be taken to be

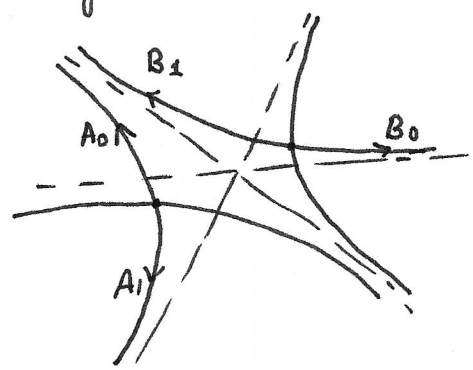
$$A_0 - A_1 \qquad 0 \leq \theta < \frac{2\pi}{3}$$

$$A_0 - A_1 + B_1 \qquad \theta = \frac{2\pi}{3}$$

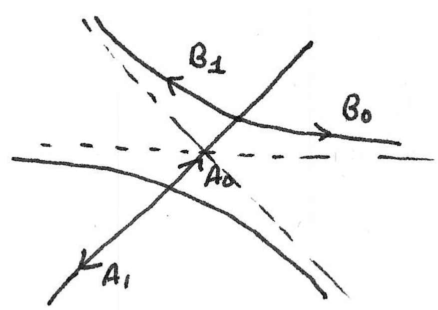
$$A_0 - A_1 - B_0 + B_1 \qquad \frac{2\pi}{3} < \theta \leq \pi$$

(Note: $Ai(\bar{z}) = \overline{Ai(z)}$, so studying $|z| \to \infty$ as $\arg(z) \in [0, \pi]$ is sufficient) - but just for fun: the formulae change at $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \ldots$

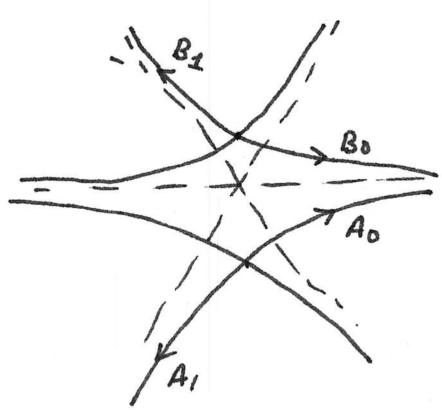
$0 \leq \theta < \frac{2\pi}{3}$; $C_1 = A_0 - A_1$ where



$\theta = \frac{2\pi}{3}$; $C_1 = A_0 - A_1 + B_1$ where



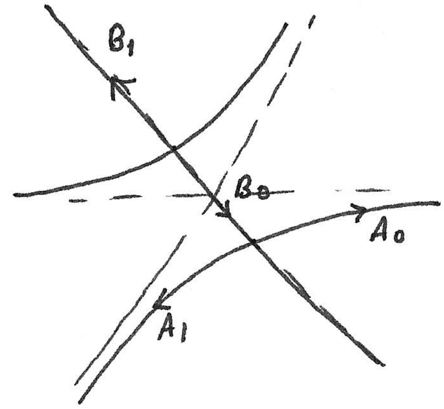
$\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$; $C_1 = A_0 - A_1 + B_1 - B_0$



$\cancel{\theta = \frac{4\pi}{3};}$

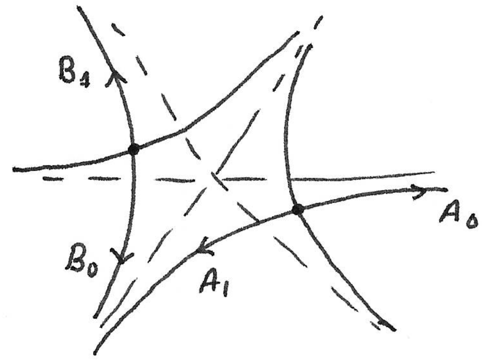
$\theta = \frac{4\pi}{3}$

$$C_1 = -A_1 - B_0 + B_1$$



$\frac{4\pi}{3} < \theta < 2\pi$

$$C_1 = -B_0 + B_1$$



§3. Using Laplace's formula, we get the following answer:

$$Ai(z) \sim \frac{z^{-1/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3} z^{3/2}\right) \quad \text{as } z \to \infty, \ \arg(z) \in (0, \pi)$$

$$\sim \frac{|z|^{-1/4}}{\sqrt{\pi}} \sin\left(\frac{2}{3}|z|^{3/2} + \frac{\pi}{4}\right) \quad \text{as } z \to \infty, \ \arg(z) = \pi.$$