Lecture 26

In the next few lectures, we will focus on geometric aspects of the mappings given by holomorphic functions. The word "mappings" is used traditionally for functions $\Omega_1 \to \Omega_2$ where $\Omega_1, \Omega_2 \subset \mathbb{R}^2$.

§1 Local inverse function theorem.

– Let $A(z) = \sum_{n=1}^{\infty} a_n z^n$ be a power series with non-zero radius of convergence, and $a_1 \neq 0$. Then $\exists!$ $B(z) = \sum_{m=1}^{\infty} b_m z^m$ such that $A(B(z)) = z$. Moreover, B has non-zero radius of convergence.

Proof. – The existence and uniqueness of B is purely an algebraic statement. Write $B(z) = b_1 z + b_2 z^2 + \cdots$ ($b_1, b_2, \ldots$ are to be determined) and impose the condition $A(B(z)) = z$:

$$a_1 (b_1 z + b_2 z^2 + \cdots) + a_2 (b_1 z + b_2 z^2 + \cdots)^2 + \cdots = z.$$

Comparing coefficients: z^1 : $a_1 b_1 = 1 \Rightarrow b_1 = \frac{1}{a_1}$ (if $a_1 \neq 0$).

$$z^2 : \quad a_1 b_2 + a_2 b_1^2 = 0 \Rightarrow b_2 = -\frac{a_2 b_1^2}{a_1}$$

$$\vdots$$

$$z^n : \quad a_1 b_n + \sum_{l=2}^{n} a_l \cdot \underbrace{(\text{coeff. of } z^n \text{ in } B(z)^l)}_{\text{only involves } b_1, \ldots, b_{n-1}} = 0$$

← assumed to be determined.

Hence $A(B(z)) = z$ determines $\{b_1, b_2, \ldots\}$ uniquely iff $a_1 \neq 0$.

Remark - Coefficient of z^n in $B(z)^\ell$

$$= \sum_{\substack{k_1, k_2, \ldots \in \mathbb{Z}_{\geq 0} \\ \boxed{\begin{array}{l} \text{s.t.} \quad k_1 + k_2 + \ldots = \ell \\ \quad k_1 + 2k_2 + 3k_3 + \ldots = n \end{array}}}} \frac{\ell!}{k_1!\, k_2! \cdots} b_1^{k_1} b_2^{k_2} \cdots =: P_{n;\ell}(\underline{b})$$

finitely many →

For $\ell \geq 2$, $P_{n;\ell}(b_1, b_2, \ldots)$ is a poly. in $b_1, \ldots, b_{n-1}$ with coeff. $\mathbb{Z}_{\geq 0}$.

$$P_{n;1}(\underline{b}) = b_n .$$

$$A(B(z)) = z \iff a_1 b_n = - \sum_{\ell=2}^{n} a_\ell P_{n;\ell}(\underline{b})$$

Convergence of $B(z)$: Let $R \in \mathbb{R}_{>0} \cup \{\infty\}$ be the radius of convergence of $A(z)$. (let us assume, for simplicity, that $a_1 = 1$).

Choose $r < R$ and $M > 0$ so that $|a_n| < \dfrac{M}{r^n}$ ($\forall\, n \geq 1$).

Set $$\widetilde{A}(x) = x - \sum_{\ell=2}^{\infty} \frac{M}{r^\ell} x^\ell = x - \frac{M}{r^2} x^2 \left(1 - \frac{x}{r}\right)^{-1}$$

$$= x - \frac{M x^2}{r(r-x)} \quad \text{for } |x| < r.$$

If $\widetilde{B}(x) = \sum_{k=1}^{\infty} \beta_k x^k$ is the (algebraic) inverse of $\widetilde{A}(x)$, then ($\beta_1 = 1$)

$$\beta_n = \sum_{\ell=2}^{n} \frac{M}{r^\ell} \overbrace{P_{n;\ell}(\underline{b})}^{\text{has non-neg. coeff's.}} \in \mathbb{R}_{\geq 0}$$

and $\underline{|b_n| \leq \beta_n \quad \forall\, n \geq 1.}$

[$\widetilde{B}(x)$ is a majorising series for $B(x)$, in that its convergence $\Rightarrow$ abs. convergence of B]

Now $y = \widetilde{B}(x)$ solves $y - \dfrac{My^2}{r(r-y)} = x$. That is,

y is the unique soln ($y=0$ for $x=0$) of the quadratic eq^n :

$$(M+r)y^2 - r(x+r)y + r^2 x = 0$$

i.e. $$y = \frac{r(x+r)}{2(M+r)}\left(1 - \left(1 - \frac{4x(M+r)}{(x+r)^2}\right)^{\frac{1}{2}}\right)$$

Using $$(1+T)^{\alpha} = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} T^n \quad \text{for } |T|<1,$$

we get $\widetilde{B}(x)$ as a power series in x with non-zero radius of convergence $\square$

§2. Examples. (1) $A(z) = z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \dots = e^z - 1$. (R.O.C. $= \infty$)

Solve for ~~W~~ $B(z)$ s.t. $A(B(z)) = z$.

$A' = e^z = 1 + A$

Taking $\dfrac{d}{dz}$ gives $$B'(z) = \frac{1}{A'(B(z))}$$
$$= \frac{1}{1+z}$$

$$\Rightarrow \quad B'(z) = 1 - z + z^2 - z^3 + \dots \quad \text{for } |z|<1.$$
$$B(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots \quad \text{for } |z|<1.$$
$$(= \log(1+z))$$

(2) $$A(z) = \sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \quad (\text{R.O.C.} = \infty)$$

$$A(B(z)) = z \quad \Rightarrow \quad A'(B(z))\, B'(z) = 1.$$

$A'(x) = \cos(x) = \sqrt{1 - A(x)^2}$ gives :

$$B'(z) = \frac{1}{\sqrt{1-z^2}} = (1-z^2)^{-\frac{1}{2}}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\cdots\left(\frac{-1}{2}-n+1\right)}{n!} z^{2n} ; \qquad |z|<1$$

$$= \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^{2n}}{4^n}$$

Hence $B(z) = \displaystyle\sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^{2n+1}}{(2n+1)\,4^n}$ for $|z|<1$.

($\arcsin(z)$ near $z=0$)

§3. Applications of Theorem §1: [Open mapping and inverse function Theorems]

Theorem : Let $\Omega \subset \mathbb{C}$ be an open, connected set, and $f : \Omega \to \mathbb{C}$ be a holomorphic, non constant function.

Let $z_0 \in \Omega$ and $N \in \mathbb{Z}_{\geq 1}$ be the order of vanishing of $f(z) - f(z_0)$ at z_0 (i.e. $f'(z_0) = \dots = f^{(N-1)}(z_0) = 0$ and $f^{(N)}(z_0) \neq 0$).

Then, there exist $\rho_1, \rho_2 \in \mathbb{R}_{>0}$ s.t. f restricted to $D(z_0;\rho_1)$ is N-to-1 function onto $D(f(z_0);\rho_2)$.

In other words: for every $w \in D(f(z_0); \rho_2)$, $w \neq f(z_0)$,

$$\left| f^{-1}(w) \cap D(z_0; \rho_1) \right| = N.$$

Note that $f^{-1}(f(z_0)) \cap D(z_0; \rho_1) = \{z_0\}$ "with mult. N".

Yet another restatement of this theorem: There exist $\rho_1, \rho_2 > 0$ and reparametrizations (biholomorphic maps = hol. + has hol. inverse)

($\mathbb{D} = D(0;1)$)

$$\begin{array}{ccc} D(z_0, \rho_1) & \xrightarrow{\simeq} & \mathbb{D} \\ f \big\downarrow & & \big\downarrow z \mapsto z^N. \\ D(f(z_0), \rho_2) & \xrightarrow{\simeq} & \mathbb{D} \end{array}$$

"locally f behaves as $z \mapsto z^N$".

Proof. For simplicity take $z_0 = 0$, $f(z_0) = 0$, so that

$f(z) = c_N z^N + \dots$ $(c_N \neq 0)$ near $z_0 = 0$.

Further assume that $c_N = 1$. (I am simply replacing $f(z)$ by $\frac{1}{c_N}\left(f(z+z_0) - f(z_0)\right)$.)

Thus, there is $R > 0$ so that $f(z) = z^N \left(1 + \sum_{\ell=1}^{\infty} c_{N+\ell} z^\ell\right)$ $\forall z \in D(0; R)$.

Now $h(z) = \sum_{\ell=1}^{\infty} c_{N+\ell} z^\ell$ also has radius of convergence R

and $h(0) = 0$. By continuity, $\exists \rho_1 > 0$ s.t. $(\rho_1 \leq R)$

$$|z| < \rho \quad \Rightarrow \quad |h(z)| < 1.$$

Hence, for $|z| < \rho$; $g(z) := (1 + h(z))^{1/N} = \sum_{\ell=0}^{\infty} \binom{1/N}{\ell} h(z)^\ell$ converges

That is, $\quad f(z) = (z \cdot g(z))^N \quad$ for $\quad z \in D(0;\rho)$.

Using Thm §1, for $A(z) = z g(z)$, we conclude that there exist $\rho_1, \rho_2 > 0$ s.t. $\quad z \cdot g(z) : D(0,\rho_1) \xrightarrow{\sim} D(0;\rho_2)$
(has holomorphic inverse)

In other words, f is a composition of

$$D(0;\rho_1) \xrightarrow{\sim} D(0;\rho_2) \xrightarrow{(\alpha \mapsto \alpha^N)} D(0;\rho_2^N). \qquad \square$$

§4. <u>Biholomorphisms ; or conformal equivalence.</u> ($\Omega_1, \Omega_2 \subset \mathbb{C}$ open, connected)

$f : \Omega_1 \to \Omega_2$ is a <u>biholomorphism</u> or <u>conformal equivalence</u> if <u>f is holomorphic</u>, (bijective) and <u>$\exists\ g : \Omega_2 \to \Omega_1$ hol.</u> such that $\underline{f \circ g = Id_{\Omega_2}}$ and $\underline{g \circ f = Id_{\Omega_1}}$.

Note : Thm §1 $\Rightarrow$ if $f : \Omega \to \mathbb{C}$ is holomorphic and $f'(z_0) \neq 0$, then f is a conformal equivalence between $D(z_0;\rho_1)$ and $D(f(z_0);\rho_2)$ for some $\rho_1, \rho_2 > 0$.

The word "conformal" refers to preservation of angles - a property crucial for navigational purposes.

§5. Preservation of angles.- Let $\Omega \subset \mathbb{C}$ be open, connected set, $f : \Omega \to \mathbb{C}$ a holomorphic function.

Proposition.— If $z_0 \in \Omega$ is such that $f'(z_0) \neq 0$, then f preserves angles incident at z_0, in the oriented sense.

Conversely, if $u, v \in C^1(\Omega, \mathbb{R})$ are such that at $(x_0, y_0) \in \Omega$, we have: (i) $(u, v) : \Omega \to \mathbb{R}^2$ preserves angles at (x_0, y_0)

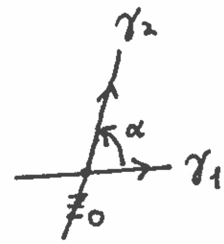
(ii) The Jacobian $u_x v_y - u_y v_x > 0$ at (x_0, y_0);

then Cauchy-Riemann eq^{n}s hold for u, v. (i.e., $u + iv : \Omega \to \mathbb{C}$ is holomorphic at z_0).

Proof.- Let $\alpha \in [0, \pi]$ and let $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to \Omega$ be given by

$$\gamma_1(t) = z_0 + t \;; \quad \gamma_2(t) = z_0 + t e^{i\alpha}.$$

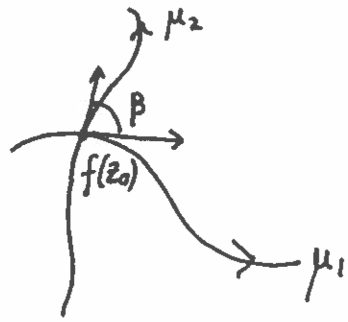
[here $\varepsilon > 0$ is small enough so that $D(z_0, \varepsilon) \subset \Omega$]



Composing with $f : \Omega \to \mathbb{C}$ gives two smooth curves through $f(z_0)$

$\mu_1, \mu_2 : (-\varepsilon, \varepsilon) \to \Omega$
$$\mu_1(t) = f(z_0 + t)$$
$$\mu_2(t) = f(z_0 + t e^{i\alpha})$$

β = angle between $\mu_1'(0)$ and $\mu_2'(0)$



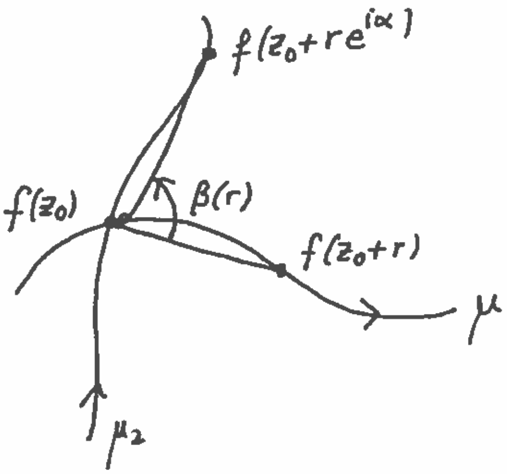
<u>To prove</u>: $\beta = \alpha$.

Proof. Let $r \in (-\varepsilon, \varepsilon)$ $(r \neq 0)$ and $\beta(r)$ = angle b/w line segment joining $f(z_0)$ & $f(z_0+r)$ and the one joining $f(z_0)$ & $f(z_0+re^{i\alpha})$

So, $\beta(r) \to \beta$ as $r \to 0$

Now,

$$\beta(r) = \arg\left(\frac{f(z_0+re^{i\alpha}) - f(z_0)}{f(z_0+r) - f(z_0)}\right)$$



$$= \arg\left(\frac{f(z_0) + f'(z_0)re^{i\alpha} + \dots - f(z_0)}{f(z_0) + f'(z_0)r + \dots - f(z_0)}\right)$$

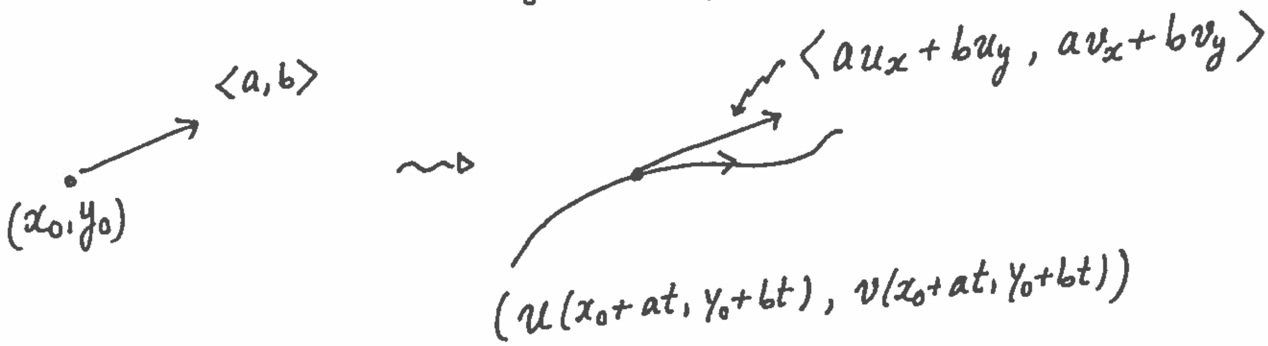
(here $\dots \in O(r^2)$)

$$= \arg\left(e^{i\alpha} + O(r)\right) \to \alpha \quad \text{as } r \to 0.$$

$\square$

Note: if $f'(z_0) = \dots = f^{(N-1)}(z_0) = 0$; $f^{(N)}(z_0) \neq 0$, then f scales the angles incident at z_0 by N.

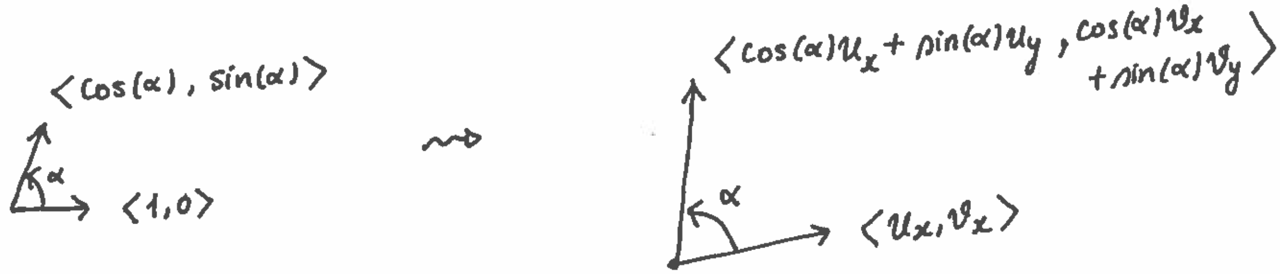
Proof of the converse: $(u, v) : \Omega \to \mathbb{R}^2$ preserve angles at (x_0, y_0). Note: $\langle u_x, u_y \rangle \neq \vec{0}$, $\langle v_x, v_y \rangle \neq \vec{0}$ at (x_0, y_0) by $u_x v_y - u_y v_x > 0$ condition.



(1) Take a pair of orthogonal vectors $\langle 1, 0 \rangle$, $\langle 0, 1 \rangle$ to get

$\langle u_x, v_x \rangle \perp \langle u_y, v_y \rangle$, i.e., $\boxed{u_x u_y + v_x v_y = 0}$

(2) Use



to get $\boxed{u_x^2 + v_x^2 = u_y^2 + v_y^2}$

Combining these two eq^{n}s, we get: either $u_x = v_y$ and $u_y = -v_x$ (orientation preserving case)

or $u_x = -v_y$ and $u_y = v_x$ (orientation reversing case)

The second case is ruled out by

$u_x v_y - u_y v_x > 0$ condition. $\square$