Lecture 27

Recall: last time we proved the following results for a holomorphic function $f : \Omega \to \mathbb{C}$ ($\Omega \subset \mathbb{C}$ an open connected set).

(1) If $z_0 \in \Omega$ is such that $f'(z_0) \neq 0$, then

(a) f preserves angles incident at z_0.

(b) f is locally (near z_0) a biholomorphism - i.e. $\exists \rho_1, \rho_2 \in \mathbb{R}_{>0}$ s.t. $f : D(z_0; \rho_1) \cong D(f(z_0), \rho_2)$.

(2) If $f'(z_0) = \cdots = f^{(N-1)}(z_0) = 0$ and $f^{(N)}(z_0) \neq 0$, then

(a) f scales by N, the angles incident at z_0.

(b) f is locally an N-to-1 covering map - i.e. $\exists \rho_1, \rho_2 > 0$ so that

and
$$\begin{tikzcd} D(z_0, \rho_1) \arrow[r, "\psi", "\sim"'] \arrow[rr, bend right, "f"'] & D(w_0, \rho_2) \arrow[r, "\alpha \mapsto \alpha^N"] & D(w_0^N, \rho_2^N) \end{tikzcd}.$$

$(w_0^N = f(z_0))$

(3) Cor: (open mapping theorem)

$f(\Omega) \subset \mathbb{C}$ is open.

Today, we will review <u>Max. Modulus and Argument principles</u>

based on Cauchy's integral formula:

$$\frac{1}{2\pi i} \int_{\circlearrowleft z_0} \frac{f(z)}{(z - z_0)^{n+1}}\, dz = \frac{f^{(n)}(z_0)}{n!} \quad \forall n \geq 0.$$

§1. <u>Prop.</u> - If $z_0 \in \Omega$ is such that $|f(z)|$ has local maximum at z_0 (i.e., $\exists\, r>0$ s.t. $|f(z_0)| \geq |f(z)|$ for every $z \in D(z_0; r)$), then f is constant.

Proof. - We will prove that f is locally constant - which by identity theorem (Ω is connected) implies that f is constant. We base our proof on the following variant of Cauchy's integral formula. obtained by setting $z = z_0 + re^{i\theta}$ $(0 \leq \theta \leq 2\pi)$

$$\boxed{f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta}$$

(mean value property of hol. fns.)

Notation - for $\rho > 0$ small enough so $\overline{D(z_0;\rho)} \subset \Omega$, set

$$M(\rho) := \operatorname{Sup}\left\{ |f(z_0 + re^{i\theta})| : 0 \leq \theta \leq 2\pi \right\}$$

Note: if $f(z_0) = 0$, there is nothing to prove; since by assumption $|f(z)| \leq |f(z_0)| = 0$ $\forall\, z \in D(z_0; r)$, we get $f(z) = 0$ $\forall\, z \in D(z_0; r)$.

Upon mult. by $e^{-i\arg(f(z_0))}$, we can assume $f(z_0) \in \mathbb{R}_{>0}$.

Let $r>0$ be as in our local max. assumption. Then $\forall\, 0<\rho<r$, we have: $M(\rho) \leq f(z_0)$. By triangle inequality and the MVP for f (mean value property):

$$M(\rho) \leq f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta$$

$$M(\rho) \leq f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta \leq M(\rho)$$

$\Rightarrow \quad f(z_0) = M(\rho) \quad \forall \rho \in (0, r)$. Combine with the following:

$$f(z_0) = \mathrm{Re}(f(z_0)) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re} f(z_0 + \rho e^{i\theta})\, d\theta$$

$$\Rightarrow \quad \int_0^{2\pi} \left(f(z_0) - \mathrm{Re} f(z_0 + \rho e^{i\theta}) \right) d\theta = 0.$$ Note: $\forall z = z_0 + \rho e^{i\theta}$,

$\mathrm{Re} f(z) \leq |f(z)| \leq M(\rho) = f(z_0)$. So, $z \mapsto f(z_0) - \mathrm{Re} f(z)$ is a non-negative, continuous fn. on the circle $C(z_0; \rho)$ whose total integral is 0. Hence, this function is identically zero, proving $\quad f(z_0) = \mathrm{Re} f(z) \quad \forall z \in D(z_0; r)$
which implies ($f(z_0) = \mathrm{Re} f(z) \leq |f(z)| \leq M(\rho) = f(z_0)$)
that $f(z) = f(z_0)$ as claimed. $\square$

§2. Remarks (a) MVP for $f \Rightarrow$ MVP for $u = \mathrm{Re} f$:

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + \rho \cos(\theta), y_0 + \rho \sin\theta)\, d\theta$$

For harmonic functions, this formula is called <u>Gauss' mean value theorem.</u> With the same proof as above, we obtain :

harmonic functions do not have local max. or local min.
(so, all critical points must be saddle points).

(b) <u>Another version of Max. Modulus principle:</u>

$f : \Omega \to \mathbb{C}$ hol.
$\overline{\Omega} \subset \mathbb{C}$ cpct
assume f extends to a cnts $\overline{\Omega} \to \mathbb{C}$.

$\Rightarrow$ $|f|$ takes its max. value at $\partial\Omega = \overline{\Omega} \setminus \Omega$.

(similarly, if $u \in C^2(\Omega; \mathbb{R})$ is harmonic, (and u extends to a cnts. fn. $\overline{\Omega} \to \mathbb{R}$) then max/min values of u occur at $\partial\Omega$.)

(c) <u>Estimates on the coefficients of Taylor Series.</u>

Let $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ be the Taylor series of f near z_0 (say, $\forall\, z \in D(z_0; r)$).

Then

$$c_n = \frac{1}{2\pi i} \oint_{C(z_0;\rho)} \frac{f(z)}{(z - z_0)^{n+1}}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho^{n+1} e^{i\theta(n+1)}}\, \rho \cdot i \cdot e^{i\theta}\, d\theta$$

$$c_n = \frac{1}{2\pi\rho^n} \int_0^{2\pi} f(z_0+\rho e^{i\theta})\, e^{-n\theta i}\, d\theta \quad ; \quad \forall \rho<r.$$

$$\Rightarrow \quad |c_n| \leq \frac{M(\rho)}{\rho^n} \qquad \text{(Cauchy estimates)}.$$

<u>Cauchy estimates are strict</u> - i.e., if $\exists\, \rho<r$ and $n\in\mathbb{N}$ s.t. $|c_n| = \frac{M(\rho)}{\rho^n}$, then $f(z)=c_n z^n$.

Proof. - For $z = z_0+\rho e^{i\theta}$, we get

$$|f(z)|^2 = f(z)\cdot\overline{f(z)} = \sum_{n,m} c_n \bar{c}_m\, \rho^{n+m} e^{i\theta(n-m)}$$

$\frac{1}{2\pi}\int_0^{2\pi} (-)\, d\theta$ on both sides to get $\left(\frac{1}{2\pi}\int_0^{2\pi} e^{i\theta l}\, d\theta = \begin{cases} 0 & \text{if } l\neq 0 \\ 1 & \text{if } l=0 \end{cases} . \right)$

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|^2\, d\theta = \sum_{n=0}^{\infty} |c_n|^2 \rho^{2n}$$

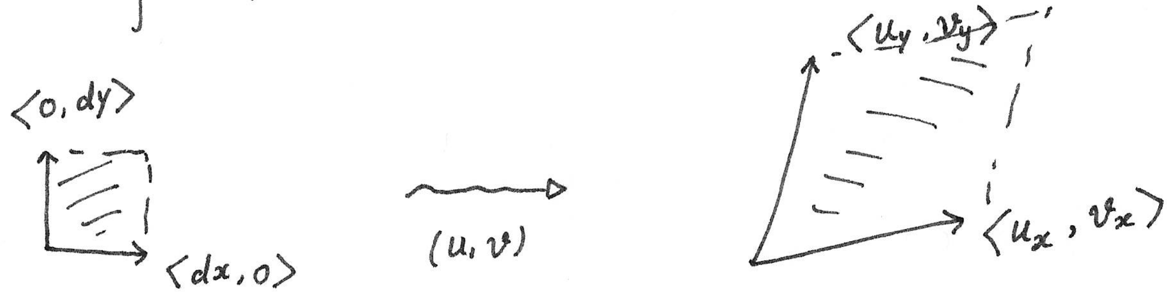
$$\Rightarrow \quad \sum_{n=0}^{\infty} |c_n|^2 \rho^{2n} \leq M(\rho)^2 .$$ The claim follows.

$\square$

§3. Area formula. - Assume $f:\Omega\to\mathbb{C}$ is one-to-one, holomorphic function (also called <u>univalent</u> in conformal geometry literature)

Then $$\boxed{\text{Area } f(\Omega) = \iint_{\Omega} |f'(z)|^2 \, dx\, dy}$$

Proof – $f = u + iv$ scales a small area element by



area = | Cross product |

$$|u_x v_y - v_x u_y|$$

$$\begin{aligned} \text{Jacobian of } (u, v) &= u_x v_y - u_y v_x \\ &= u_x^2 + u_y^2 \quad \text{by Cauchy-Riemann eq}^{\underline{n}}\text{s} \\ &= |f'(z)|^2 . \end{aligned}$$ $\square$

Cor. Assume $\Omega = D(0; R)$ and $f(z) = \sum_{n=0}^{\infty} c_n z^n$ on Ω.

Then $$\text{Area } f(\Omega) = \pi \sum_{n=1}^{\infty} n |c_n|^2 R^{2n}$$

Pf: $f'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1}$, $|f'(z)|^2 = \sum_{n,m=1}^{\infty} n m \, c_n \bar{c}_m z^{n-1} (\bar{z})^{m-1}$

So, $$\text{Area}(f(\Omega)) = \int_0^{2\pi}\int_0^R \sum_{n,m=1}^{\infty} nm\, c_n \overline{c_m}\, \rho^{n+m-2} e^{i\theta(n-m)} \cdot \rho\, d\rho\, d\theta$$

$$= \sum_{n=1}^{\infty} n^2 |c_n|^2 \cdot \cancel{2}\pi \cdot \frac{R^{2n}}{\cancel{2}n}$$

$$= \pi \sum_{n=1}^{\infty} n \cdot |c_n|^2 R^{2n}.$$

$\square$

§4. Argument principle.-

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \begin{array}{c}\text{Number of}\\ \text{zeroes}\\ \text{of } f\end{array} - \begin{array}{c}\text{Number of}\\ \text{poles}\\ \text{of } f\end{array} \in \mathbb{Z}.$$

(within C).

here, C is a contour (piecewise C^1, simple, closed, counterclockwise oriented path - reminder!)

f is a meromorphic fn. on a domain Ω containing C & interior(C).

f has no zeroes or poles on C.

<u>Proof.</u>- If $\alpha \in$ interior(C) and $f(z)$ has the following form near α : $f(z) = (z-\alpha)^N \cdot \phi(z)$, $(\phi(\alpha) \neq 0)$

($N \in \mathbb{Z}$; $N>0$: zero, $N<0$: pole) then

$$\frac{1}{2\pi i}\int_{\circlearrowleft \alpha}\frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_{\circlearrowleft \alpha}\Big(\frac{N}{z-\alpha}+\underbrace{\frac{\phi'(z)}{\phi(z)}}_{0 \ (\phi \text{ is hol. at } \alpha)}\Big)\,dz$$

$$= N.$$

The proof follows from deforming C to a disjoint union of small circles around poles of $\frac{f'}{f}$ (same as zeroes/poles of f) in interior(C). $\square$

Remarks. – (a) The formula $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-\alpha}\,dz$ counts number of solns. to $f(z)=\alpha$ within C (assume f is hol. on interior(C)).

(b) $$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \underbrace{\frac{1}{2\pi i}\int_{\gamma = f(C)}\frac{dw}{w}}_{\text{called winding number of } \gamma \text{ around } 0.} \qquad (w=f(z))$$

§5. Regularity of harmonic functions

If $u \in C^2(\Omega; \mathbb{R})$ is a harmonic fn. on a simply connected domain Ω, then $u = \mathrm{Re} f$ for some $f : \Omega \to \mathbb{C}$ hol.

Hence $u \in C^\infty(\Omega; \mathbb{R})$, or even more - u is analytic.

(the last assertion is local - so the simply-connected hypothesis can be dropped)

Proof.- Set $g(z) = u_x(x,y) - i\, u_y(x,y)$ $\quad (z = x + iy)$.

$g \in C^1(\Omega; \mathbb{C})$ and Cauchy-Riemann eq^{n}s hold

$$\Big((\mathrm{Re}\, g)_x = u_{xx} \underset{(u \text{ is harmonic})}{=} -u_{yy} = (\mathrm{Im}\, g)_y \; ; \; (\mathrm{Re}\, g)_y = u_{xy} = +u_{yx} = -(\mathrm{Im}\, g)_x . \Big)$$

So, $g : \Omega \to \mathbb{C}$ is holomorphic. ~~Set~~

Choose $z_0 \in \Omega$ and define

$$f(z) = \underbrace{\int_{z_0}^{z} g(s)\, ds}_{\text{independent of path } z_0 \to z \text{ (in } \Omega)} + u(x_0, y_0)$$

since g is hol. & Ω is simply-connected.

$$\mathrm{Re} f(z) = \int_{z_0}^{z} u_x(s)\, ds + u(x_0, y_0)$$

$= u(z)$ by fund. thm. of calculus.

$\square$