Lecture 28

Recap: $\underset{\text{holomorphic}}{f : \Omega \to \mathbb{C}}$; $\underset{\text{open, connected}}{\Omega \subset \mathbb{C}}$. In last two lectures, we reviewed the following properties of f.

(1) $f'(z_0) \neq 0$ $(z_0 \in \Omega)$ $\Rightarrow$ (a) f preserves angles incident at z_0.

(b) $\exists\, \rho_1, \rho_2 > 0$ s.t. $f : D(z_0; \rho_1) \cong D(f(z_0), \rho_2)$

(2) $f(\Omega) \subset \mathbb{C}$ is open.

(3) If $|f| : \Omega \to \mathbb{R}_{\geq 0}$ has a local max. at $z_0 \in \Omega$, then f is constant.

(4) Let C be a contour in Ω so that f is mero. on the interior of C. Then:

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz = N_f(C) \in \mathbb{Z}$$

$$= \underset{\substack{\uparrow \\ \#\text{ zeroes of } f \\ \text{within } C}}{Z_f(C)} - \underset{\substack{\uparrow \\ \#\text{ poles of } f \\ \text{within } C}}{P_f(C)}$$

(counted with multiplicity)

$$= \frac{1}{2\pi i} \int_{\gamma = f(C)} \frac{dw}{w} = \text{winding number of } \gamma \text{ around } 0.$$

§1. Application – counting zeroes of f within C.

(<u>Rouché's Theorem</u>)

Let C be a contour in $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ hol. fn. ; $g : \Omega \dashrightarrow \mathbb{C}$ mero. fn.

[$\Omega \subset \mathbb{C}$ open, connected s.t. $C \cup \text{interior}(C) \subset \Omega$]

s.t. $fg : \Omega \to \mathbb{C}$ is hol.

Assume f and g have no zeroes or poles on C and

(*) $g(C)$ lies in a simply-connected subset of $\mathbb{C} \setminus \{0\}$.

[want : winding number of $g(C)$ around 0 to be 0.]

e.g. if $g(z) \in D(1;1)$ $\forall z \in C$.

Then <u>$f(z) = 0$ and $f(z)g(z) = 0$</u> have <u>the same number of</u> <u>solutions</u> <u>in Interior(C)</u>.

Proof :

$$\frac{1}{2\pi i}\int_C \frac{(f(z)g(z))'}{f(z)g(z)}\,dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz + \underbrace{\cancel{\frac{1}{2\pi i}\int_C \frac{g'(z)}{g(z)}\,dz}}_{0}$$

$\square$

e.g.

$$p(z) = z^{10} - 8z + 1 = \underbrace{z^{10}}_{f(z)}\underbrace{(1 - 8z^{-9} + z^{-10})}_{g(z)}$$

If $|z| = 2$ (say), $\left| g(z) - 1 \right| \leq \frac{8}{2^9} + \frac{1}{2^{10}} < 1$. Hence $p(z) = 0$ have as many solutions as $z^{10} = 0$ within $D(0;2)$, i.e. 10.

Thus Rouché's Thm. gives another proof of the fundamental theorem of algebra. To illustrate that it is stronger- let us work with the same example.

$$P(z) = z^{10} - 8z + 1 = \underbrace{-8z}_{f(z)} \underbrace{\left(1 - \frac{1}{8}z^{9} - \frac{1}{8}z^{-1}\right)}_{g(z)}$$

$$|z| = 1 \implies |g(z) - 1| \leq \frac{1}{8} + \frac{1}{8} < 1.$$

$f(z)=0$ has only 1 soln. in $D(0;1)$

So, $P(z) = 0$ has 1 soln with $|\cdot| < 1$.

The rest 9 are in $Ann(0; 1<2)$ $(= \{z : |z| \in (1,2)\})$.

[More on this next semester.]

§2. Conformal equivalences / automorphisms.

For $\Omega \subset \mathbb{C}$ open, connected, let $Aut(\Omega) = \{ f : \Omega \to \Omega \mid f$ is a biholomorphism $\}$

Theorem. - $Aut(\mathbb{C}) = \{ z \mapsto az + b : a, b \in \mathbb{C} ; a \neq 0 \}$

i.e. if $f : \mathbb{C} \to \mathbb{C}$ is one-to-one and holomorphic, then $\exists\, a, b \in \mathbb{C}$, $a \neq 0$, s.t. $f(z) = az + b$.

Proof. As $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, ∞ is an isolated singularity of f. We claim that it cannot be essential.

Assume it is true (i.e. ∞ is either removable or a pole of order $n \geq 1$). If ∞ is removable, then by Liouville's theorems f is constant, contradicting that it was 1-1.

If it is a polynomial of degree n (the case when ∞ is a pole of order $n \geq 1$) then n must be 1 (by FTA) and we are done.

<u>∞ is not an essential singularity of f</u> : The proof is based on <u>Casorati-Weierstrass theorem</u>, which states that: if ∞ is an essential singularity of f, then $\forall R>0$, $f(\{|z|>R\}) \subset \mathbb{C}$ is dense. As $f(\{|z|<R\})$ is open, we get a contradiction to injectivity of f : $f(\{|z|<R\}) \cap f(\{|z|>R\}) \neq \phi$.

<u>Proof of Casorati-Weierstrass Theorem</u> : Assume $\exists R>0$ s.t $f(\{|z|>R\})$ is not dense - i.e. $\exists$ $w \in \mathbb{C}$ and $r>0$ s.t. $f(\{|z|>R\}) \cap D(w;r) = \phi$.

i.e. $\forall z \in \Omega$, $|f(z)-w| > r$

$(\Omega = \{u \in \mathbb{C} : |u|>R\})$

$\Rightarrow g(z) := \dfrac{1}{f(z)-w} : \Omega \to \mathbb{C}$ has removable sing. at ∞.

If $g(z)$ has a zero of order N at ∞ $(N \geq 0)$,

then $f(z) = \dfrac{1}{g(z)} + w$ has a pole of ord. N at ∞,

contradicting the fact that ∞ was assumed to be essential $\square$

§3. $$\mathrm{Aut}(\mathbb{D}) = \left\{ z \mapsto e^{i\phi} \frac{z+\alpha}{1+\bar{\alpha} z} \;:\; \begin{array}{l} \alpha \in \mathbb{D} \\ \phi \in \mathbb{R}/_{2\pi\mathbb{Z}} \end{array} \right\}$$

(Schwarz' Lemma). $[\mathbb{D} = D(0;1) = \{z : |z| < 1\}.]$

Let $f : \mathbb{D} \to \mathbb{D}$ be s.t. $f(0) = 0$. Then (i) $|f(z)| \leq |z|$ $\forall z \in \mathbb{D}$.

(ii) If $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D}$, then $f(z) = \lambda z$ for some $\lambda \in S^1$ (i.e. $|\lambda| = 1$).

Proof.- Define $$g(z) := \begin{cases} f(z)/z & ; z \neq 0 \\ f'(0) & ; z = 0 \end{cases}$$ $g : \mathbb{D} \to \mathbb{C}$ holomorphic

and $|g(z)| < \dfrac{1}{|z|}$. For each $r \in (0,1)$, the function $|g|$ is bounded by $\dfrac{1}{r}$ on the circle $C(0;r)$. By max. modulus principle, $|g| < \dfrac{1}{r}$ on $D(0;r)$ - i.e. $|f(z)| < \dfrac{|z|}{r}$ $\forall z$ s.t. $|z| \leq r$.

In the limit $r \to 1^-$, we get $|f(z)| \leq |z|$.

If $|f(z_0)| = |z_0|$, then $|g|$ attains its $\underset{(1)}{\text{max.}}$ at z_0,

hence $g(z) =$ constant of modulus 1, say λ

$\Rightarrow f(z) = \lambda \cdot z$ $\square$

<u>Exercise:</u> Verify that $\forall\, a \in \mathbb{D}$, $T_a(z) = \dfrac{z-a}{1-\bar{a}z}$ maps

$\mathbb{D}$ to itself., and is a biholomorphism.

Now, let $T \in \mathrm{Aut}(\mathbb{D})$, i.e. $T : \mathbb{D} \to \mathbb{D}$ is (one-to-one, surjective)

biholomorphism. Let $a = T(0)$.

Replacing T by $S = T_a \circ T$, we get $S : \mathbb{D} \xrightarrow{\sim} \mathbb{D}$
$S(0) = 0$.

As both S and S^{-1} satisfy the hypothesis of Schwarz Lemma,

we get:

$$|S(z)| \leq |z| \qquad (\forall\, z, w \in \mathbb{D})$$
$$|S^{-1}(w)| \leq |w|$$

Set $w = S(z)$ to get
$|z| \leq |S(z)|$

Hence $|S(z)| = |z|$ $\forall\, z \in \mathbb{D}$, implying $S(z) = \lambda z$

for some $\lambda \in S^1$. $\square$