Lecture 29

Recall: we defined, for $\Omega \subset \mathbb{C}$, (open & connected) $\quad \mathrm{Aut}(\Omega) = \{ f : \Omega \xrightarrow{\sim} \Omega$ bijective and holomorphic $\}$

$\mathrm{Aut}(\Omega)$ = group of conformal automorphisms of Ω.

In the previous lecture, we showed:

$$\mathrm{Aut}(\mathbb{C}) = \{ z \mapsto az+b \; : \; a, b \in \mathbb{C}, \, a \neq 0 \}$$

$$\mathrm{Aut}(\mathbb{D}) = \left\{ z \mapsto e^{i\phi} \frac{z-\alpha}{1-\bar{\alpha} z} \; : \; \begin{array}{l} \alpha \in \mathbb{D} \\ \phi \in \mathbb{R} \bmod 2\pi\mathbb{Z} \end{array} \right\}$$

§1. Möbius transformations

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, let $M_A : \mathbb{C} \dashrightarrow \mathbb{C}$

be given by $\boxed{M_A(z) = \dfrac{az+b}{cz+d}}$.

Lemma.

(i) $M_{\lambda A} = M_A$ for any $\lambda \in \mathbb{C}$.

(ii) $M_{A_1} \circ M_{A_2} = M_{A_1 A_2}$

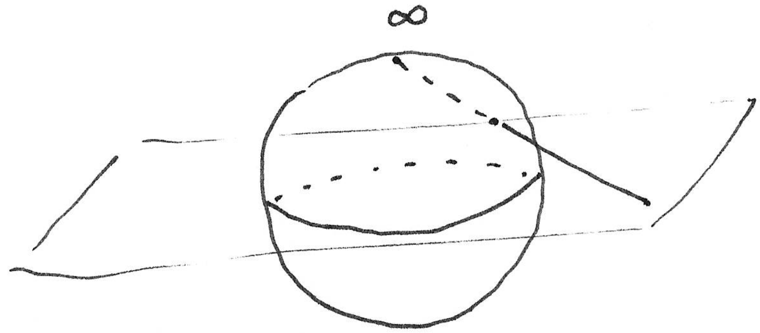
(iii) $M_{Id} : z \mapsto z$

(Proof. left as an easy exercise.)

Note: $$M_A'(z) = \frac{(cz+d)a - (az+b)\cdot c}{(cz+d)^2} = \frac{\det(A)}{(cz+d)^2}$$

Hence, if $\det(A) = 0$, $M_A(z)$ is a constant.
if $\det(A) \neq 0$, by (ii) & (iii) above $M_A^{-1} = M_{A^{-1}}$.

§2. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the Riemann Sphere.



Picture of Riemann Sphere and stereographic projection.

$\mathrm{Aut}(\hat{\mathbb{C}})$ consists of all bijections $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ s.t. $f : \mathbb{C} \dashrightarrow \mathbb{C}$ is meromorphic.

<u>Prop.</u> $\mathrm{Aut}(\hat{\mathbb{C}}) = \left\{ M_A : A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ s.t. } ad - bc \neq 0 \right\}$.

Proof.- If $f : \mathbb{C} \dashrightarrow \mathbb{C}$ is meromorphic and extends to infinity so that $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a bijection, then:

Case 1 : $f(\infty) = \infty$. Then $f : \mathbb{C} \to \mathbb{C}$ is a conformal eq.

Hence $\underset{(a \neq 0)}{f(z) = az + b} = M_{\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}}(z)$.

Case 2 : $f(\infty) \neq \infty$. Then $\exists!\ \beta \in \mathbb{C}$ s.t. $f(\infty) = \beta$.

Let $M(z) = \dfrac{1}{z - \beta} = M_{\begin{bmatrix} 0 & 1 \\ 1 & -\beta \end{bmatrix}}(z)$. Then

$M \circ f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is again in $\mathrm{Aut}(\hat{\mathbb{C}})$ and sends $\infty \mapsto \infty$. We are done by the previous argument.

$\square$

§3. Examples (elementary Möbius transformations)

Scaling: $\sigma_p(z) = pz \quad (p \in \mathbb{C} \setminus \{0\})$

Translation: $\tau_x(z) = z + x \quad (x \in \mathbb{C})$

Inversion: $I(z) = \dfrac{1}{z}$

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{cz+d}, \quad \text{if } c \neq 0$$

$$= \frac{a}{d} z + \frac{b}{d}, \quad \text{if } c = 0$$

Hence, $$M_A = \tau_{\frac{a}{c}} \circ \sigma_{-ad+bc} \circ I \circ \tau_d \circ \sigma_c \quad (\text{if } c \neq 0)$$

$$= \tau_{b/d} \circ \sigma_{a/d} \quad (\text{if } c = 0)$$

<u>Every Möbius transformation can be written as a composition of elementary ones.</u>

§4. <u>Fixed points</u>. (i.e., solutions to $M(z) = z$). Every Möbius transformation has 1, 2 or infinitely many ($M(z) = z$, $\forall z \in \hat{\mathbb{C}}$) fixed points:

Proof: $az + b = z(cz+d)$ is quadratic iff $c \neq 0$

is linear iff $c = 0$ and $a \neq d$

If $c = 0$ and $a = d = 1$ (scale to 1), $M_A(z) = z + b = z$ has 1 soln ($z = \infty$) $\Leftrightarrow b \neq 0$

$\square$

§5. <u>Theorem</u>.- Given three distinct points $\alpha, \beta, \gamma \in \hat{\mathbb{C}}$, there is unique $M \in Aut(\hat{\mathbb{C}})$ s.t.

$$M(\alpha) = 0$$
$$M(\beta) = 1$$
$$M(\gamma) = \infty$$

Proof.- Uniqueness follows from the result of §4 above. for existence, we consider the following 2 cases:

(i) $\infty \notin \{\alpha, \beta, \gamma\}$. Then $M(z) = \dfrac{\beta - \gamma}{\beta - \alpha} \, \dfrac{z - \alpha}{z - \gamma}$

(ii) Say $\gamma = \infty$; $\alpha, \beta \in \mathbb{C}$ $(\alpha \neq \beta)$, $M(z) = \dfrac{z - \alpha}{\beta - \alpha}$

$\square$

Defn.- <u>Cross ratio of 4 points on $\hat{\mathbb{C}}$</u>: If $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$, (4 distinct points)

then $\boxed{[z_1 : z_2 : z_3 : z_4] := M(z_4)}$ where $M \in Aut(\hat{\mathbb{C}})$ is the unique Möbius transformation sending

$$z_1 \mapsto 0$$
$$z_2 \mapsto 1$$
$$z_3 \mapsto \infty$$

<u>Cor</u>. Let $\{z_1, z_2, z_3, z_4\}$; $\{w_1, w_2, w_3, w_4\}$ be two sets of 4 distinct points on $\hat{\mathbb{C}}$. Then, $\exists\, M \in Aut(\hat{\mathbb{C}})$ s.t. $M(z_j) = w_j \;\; \forall\, 1 \leq j \leq 4$

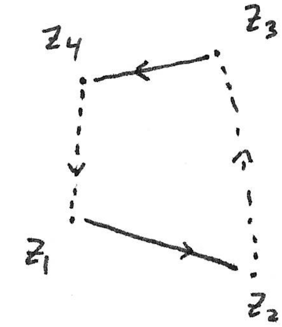
$$\iff \quad [z_1 : z_2 : z_3 : z_4] = [w_1 : w_2 : w_3 : w_4].$$

e.g. assume $\infty \notin \{z_1, z_2, z_3\}$, so that

$$M(z) = \frac{z_2 - z_3}{z_2 - z_1} \; \frac{z - z_1}{z - z_3} \, . \quad \text{By definition}$$

$$[z_1 : z_2 : z_3 : z_4] = \frac{z_2 - z_3}{z_2 - z_1} \; \frac{z_4 - z_1}{z_4 - z_3}$$

$$= \frac{(z_2 - z_3)(z_4 - z_1)}{(z_1 - z_2)(z_3 - z_4)}$$



..... Numerator

— denominator

§6. Preservation of circles on $\hat{\mathbb{C}}$

Easy exercise: under stereographic projection $\rho : \hat{\mathbb{C}} \setminus \{\infty\} \to \mathbb{C}$

Circles on $\hat{\mathbb{C}}$ go to either circles, or lines in $\mathbb{C}$

Circles passing through ∞. $\longleftrightarrow$ Lines in $\mathbb{C}$

Lengthy - but straight-forward calculation checks the effect of elementary Möbius transformations on lines and circles on $\mathbb{C}$:

Notation: $C(\alpha; r) = \{z : |z - \alpha| = r\}$

$L(w; a) = \{z : \mathrm{Re}(zw) = a\}$ ← line with equation $Ax + By = a$; if $w = A - Bi$.
$(w \neq 0)$

Translation : $\tau_x(C(\alpha;r)) = C(\alpha+x;r)$

$(x \in \mathbb{C})$ $\quad \tau_x(L(w;a)) = L(w;\ a + \mathrm{Re}(xw))$

Scaling : $\sigma_p(C(\alpha;r)) = C(p\alpha,\ |p|\cdot r)$

$(p \in \mathbb{C}\setminus\{0\})$ $\quad \sigma_p(L(w;a)) = L(\frac{w}{p};\ a)$

Inversion :

$$I(C(\alpha;r)) = \begin{cases} L(\alpha;\frac{1}{2}) & \text{if } |\alpha| = r \text{ (i.e. } 0 \in C(\alpha;r)) \\ C\left(\frac{\bar{\alpha}}{|\alpha|^2-r^2};\ \frac{r}{||\alpha|^2-r^2|}\right) & \text{if } |\alpha| \neq r. \end{cases}$$

$$I(L(w;a)) = \begin{cases} L(\bar{w};0) & \text{if } a = 0\text{, i.e. line through } 0 \\ C(w;|w|) & \text{if } a = \frac{1}{2} \end{cases}$$

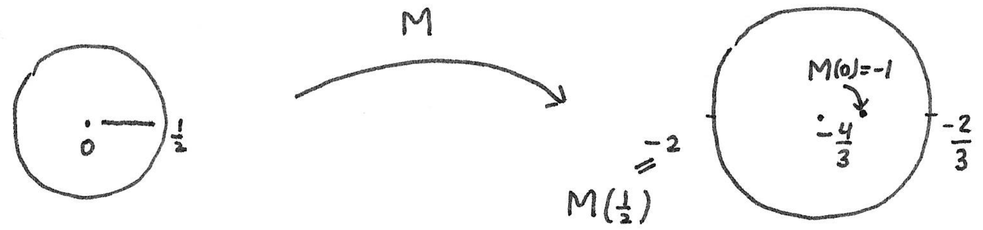
note : if $a \neq 0$, $L(w;a) = L(\frac{w}{2a};\frac{1}{2})$ - so the formula above covers all cases.

$$\boxed{\text{Möbius transformations map circles in } \hat{\mathbb{C}} \text{ to circles in } \hat{\mathbb{C}}}.$$

e.g. $M(z) = \dfrac{1}{z-1}$. Let $C = C(0;\frac{1}{2})$.

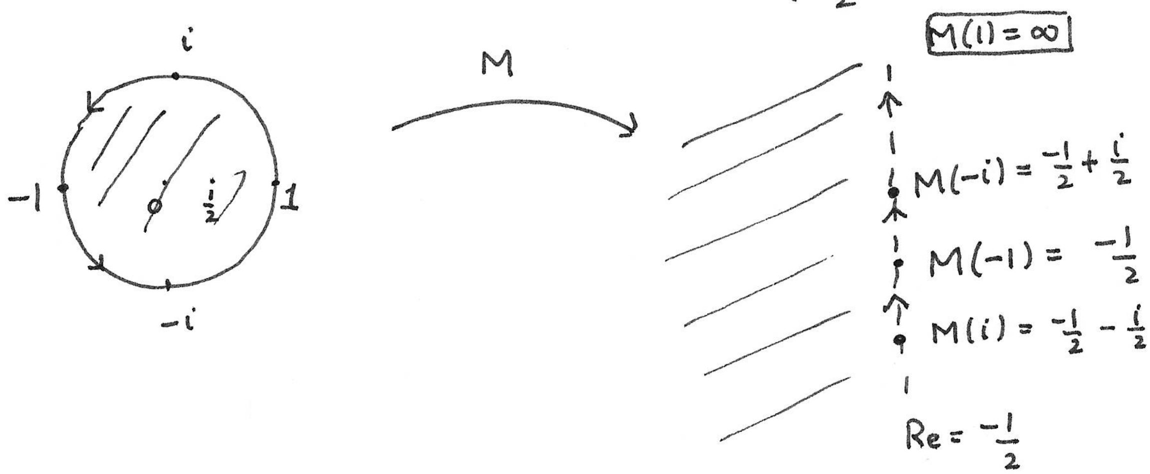
$M = I \circ \tau_{-1}$: $C = C(0;\frac{1}{2}) \longmapsto I(C(-1;\frac{1}{2}))$

$= C(-\frac{4}{3};\frac{2}{3})$



$$\left(M(z) = \frac{1}{z-1} \right)$$

$$M(C(0;1)) = I(C(-1;1))$$
$$= L(-1;\tfrac{1}{2}) = \{z : \mathrm{Re}(-z) = \tfrac{1}{2}\}$$
$$= \{\tfrac{-1}{2} + it : t \in \mathbb{R}\}$$



Cayley Transform : $C(z) = \dfrac{z-i}{z+i}$ is a Möbius transformation setting up conformal equivalence between the upper half plane and the unit disc.